(1) Let $R$ be a ring with unity.
   
   (a) Show that the set of units $R^X = \{ r \in R | \exists r^{-1} \in R, rr^{-1} = r^{-1}r = 1 \}$ in $R$ forms a group (under multiplication).
   
   (b) Show that the group of units in $\mathbb{Z}_{10}$ is a cyclic group of order 4 (thus isomorphic to $\mathbb{Z}_4$).

(2) Let $R$ be a ring. $a \in R$ is an idempotent if $a^2 = a$.
   
   (a) Show that the set of idempotents in a commutative ring is closed under multiplication.
   
   (b) Determine the set of idempotents in $\mathbb{Z}_6 \times \mathbb{Z}_{12}$.

(3) Find the following remainders:
   
   (a) $3^{47}$ when divided by 23
   
   (b) $11^{1202}$ when divided by 36

(4) Let $R$ be an integral domain.
   
   (a) Show that $R$ must be of characteristic 0 or $p \in \mathbb{N}$ where $p$ is a prime.
      
      Recall: For a nontrivial ring $R$ where every element is of finite (additive) order, the ring $R$ is of characteristic $n$ if $n$ is the least positive integer such that $r + r + \cdots + r = n \cdot r = 0$ for all $r \in R$. If there is an element of $R$ that is not of finite (additive) order, then $R$ is of characteristic 0.
      
   (b) Show that $\deg(fg) = \deg(f) + \deg(g)$ for all non-zero polynomials $f, g \in R[X]$. Show by example that this isn’t the case in $\mathbb{Z}_6[X]$ (even when the product is non-zero).

(5) Suppose $R$ is a ring with unity, $S$ is an integral domain and $\phi : R \to S$ is a ring homomorphism such that $\phi(R) \neq \{0\}$. Show that $\phi(1)$ is the unity element in $S$. 