(1) Consider $f(x) = x^3 + 2x + 4$ and $g(x) = x + 2$ as a polynomials in $\mathbb{Z}_5[X]$. Find polynomials $q, r$ in $\mathbb{Z}_5[X]$ such that $f = gq + r$ and $r \in \mathbb{Z}_5$.

(2) Let $I, J$ be ideals of a ring $R$.
   (a) Show that the $I \cap J$ is an ideal of $R$.
   (b) Show that $I + J = \{a + b|a \in I, b \in J\}$ is an ideal of $R$.

(3) Let $F$ be a field.
   (a) Show that the only ideals of $F$ are $\{0\}$ and $F$.
   (b) Show that $F$ is isomorphic to $F/\{0\}$.
   (c) Let $R$ be a ring and suppose $\phi : F \to R$ is an onto ring homomorphism. Show that if $\phi(F) \neq \{0\}$ then $F \cong R$.

(4) Prove that...
   (a) $x^2 + 2x + 2$ is reducible in $\mathbb{C}[X]$, but irreducible in $\mathbb{R}[X]$.
   (b) $x^2 + 1$ is reducible in $\mathbb{Z}_5[X]$, but irreducible in $\mathbb{Z}_7[X]$.

(5) Let $R$ be a commutative ring. We say that $a \in R$ is nilpotent if $a^n = 0$ for some positive integer $n$.
   (a) Show that the set of all nilpotent elements in $R$ forms an ideal (called the nilradical of $R$).
   (b) Show that if $N$ is the nilradical of $R$ then the nilradical of $R/N$ is the trivial ideal $\{0 + N\}$. 