NO CALCULATORS NOR NOTES ARE ALLOWED.

(1) (3 pts each) Fill-in the blanks (after each ellipses) with words:

(a) Let $X$ be a nonempty set in a universe $U$. The set of elements that are in $U$, but not in $X$ is called the \underline{complement} of $X$.

(b) Let $X$ be a nonempty set. A collection of nonempty subsets of $X$ whose union is ... $X$ and that are pairwise ... \underline{disjoint} is called a partition of $X$.

(c) A conditional proposition (aka an implication) is a proposition of the form ...

... \underline{if $p$ then $q$} where $p, q$ are propositions.

(d) Let $n \in \mathbb{N}$. The \underline{pigeonhole principle} states that if more than $n$ objects are placed into $n$ categories then at least one category has more than one object.

(e) A \underline{proof by contradiction} shows that a proposition $p$ is true by showing that $\neg p$ implies $q \land \neg q$ for some proposition $q$.

(f) Let $k, n \in \mathbb{Z}$. $n$ is \underline{divisible by $k$} if there exists $m \in \mathbb{Z}$ such that $n = mk$.

(g) The \underline{power set} of a set $X$ is the set of all subsets of $X$. It's denoted $\mathcal{P}(X)$.
(2) (3 pts each) True or False? No justification required!

(a) For \( A = \left\{ \left( \frac{1}{n}, n+1 \right) \mid n \in \mathbb{N} \right\} \) it follows that \( \bigcup_{I \in A} I = \mathbb{R}^+ \)
and \( \bigcap_{I \in A} I = (1, 2) \).

[True]

(b) For any propositions \( p, q \), the compound propositions \( p \Rightarrow q \) and \( \neg p \lor q \) are logically equivalent.

[True]

(c) \( \forall k \in \mathbb{N}, \exists S \in \mathcal{P}(\{1, 2, \ldots, k\}) \) such that \( S \neq \emptyset \) and \( \forall x, y \in S, x - y \) is odd.

[False]

(d) If \( 4 + 2 = 42 \) then Pluto is a planet.

[True (by default)]

(e) If 37 distinct numbers are selected from \( \{1, 2, \ldots, 50\} \) then there will be a pair whose sum is 75.

[False]

(f) \( \forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n-1)}{2} \).

[False]
(3) (3 pts each) Let $X = \{1, 2, 3\}$.

(a) Write all the elements of the set $S = \{(a, b) \in X \times X | b \geq a\}$ within set brackets.

\[ \{ (1,1), (1,2), (1,3), (2,2), (2,3), (3,3) \} \]

(b) Write down all elements of the set $\mathcal{P}(X)$ within set brackets.

\[ \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \} \]

(c) List all partitions of $X$ using correct set bracket notation. For example, $\{\{1\}, \{2, 3\}\}$ is one partition of $X$ (and should be included on your list).

\[ \{ \{X\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \} \]
(4) (3 pts) Negate the following statement:

\[ \forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, \forall z \in \mathbb{R}, \ z > y \Rightarrow \frac{1}{z} < x. \]

\[ \exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, \exists z \in \mathbb{R}, \ z > y \land \frac{1}{z} \geq x. \]

(5) (3 pts) Jack gets paid 26 times a year. Explain (in detail) why it is guaranteed that during one month Jack gets paid at least 3 times. Please use complete and grammatically correct sentences.

Since \( 26 \geq 1 + 12 \cdot 2 = 25 \)

& there are 12 months

by the strong form of pigeonhole principle there must be a month during which Jack gets paid at least three times.

[Could also argue by contradiction!]

[Although it seems that 26 payments are insufficient to meet the requirement of paying at least 3 times in a month, the argument is correct.]
(6) Indicate clearly the problem to be graded. No extra credit for completing more than 3.

(a) (7 pts) \( \forall n \in \{3, 4, \ldots\}, (2n)! > 5^n \).

Hint: \( 6! = 720 \) and \( 5^3 = 125 \).

Suppose \( n = 3 \). Then

\[(2n)! = 6! = 720 \quad \text{and} \quad 5^n = 5^3 = 125.\]

Thus the base case holds since \( (2n)! > 5^n \) for \( n = 3 \).

Let \( n \geq 3 \) be an integer and assume \( (2n)! > 5^n \). Then

\[\left[2(n+1)\right]! = (2n+2)(2n+1)(2n)! > 5 \cdot 5^n = 5^{n+1}\]

Since \( n \geq 3 \) implies \( (2n+2)(2n+1) > 5 \),

That completes the proof by induction.
(b) (7 pts) \( \forall n \in \mathbb{N}, \sum_{k=1}^{n} \frac{1}{k^2 + k} = \frac{n}{n+1} \).

Suppose \( n = 1 \). Then \( \sum_{k=1}^{1} \frac{1}{k^2 + k} = \frac{1}{2} = \frac{1}{1+1} \).

Thus the base case holds. Let \( n \in \mathbb{N} \).

Assume \( \sum_{k=1}^{n} \frac{1}{k^2 + k} = \frac{n}{n+1} \). Then

\[
\sum_{k=1}^{n+1} \frac{1}{k^2 + k} = \left( \sum_{k=1}^{n} \frac{1}{k^2 + k} \right) + \frac{1}{(n+1)^2 + (n+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}
\]

\[
= \frac{1}{n+1} \left[ \frac{n^2 + 1}{n+2} \right]
\]

\[
= \frac{1}{n+1} \left[ \frac{2n^2 + 2n + 1}{n+2} \right]
\]

\[
= \frac{(n+1)^2}{(n+1)(n+2)}
\]

\[
= \frac{(n+1)}{(n+1)+1}
\] \( \square \)

(c) (7 pts) \( \forall n \in \mathbb{N}, \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) = n + 1 \).

Let \( n \in \mathbb{N} \).

Then

\[
\prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) = \prod_{k=1}^{n} \frac{k+1}{k}
\]

\[
= \left( \frac{2}{1} \right) \left( \frac{3}{2} \right) \cdots \left( \frac{n+1}{n} \right)
\]

\[
= (n+1) \bigg/ \frac{1}{x} \bigg/ \frac{n}{x-1} \bigg/ \cdots \bigg/ \frac{2}{2}
\]

\[
= (n+1)
\] \( \square \)

[OR: By induction!]
(d) (7 pts) For all sets $X, Y$, $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$.

For 3 bonus points show the reverse inclusion doesn’t hold by counterexample (only applies if you attempt this proof).

Let $X, Y$ be sets and $A \in \mathcal{P}(X \cup Y)$. If $A \subseteq X$, then $A \subseteq X \cup Y$. If $A \subseteq Y$, then $A \subseteq X \cup Y$. Either way $A \subseteq X \cup Y$, so $A \in \mathcal{P}(X \cup Y)$. Thus $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$. $\square$

Bonus: $X = \{1, 3\}$, $Y = \{3, 5\}$. Then $X \cup Y \in \mathcal{P}(X \cup Y)$, but $X \subseteq \mathcal{P}(X \cup Y)$.

(e) (7 pts) For all $x, y \in \mathbb{R}$, if $x + e \geq y$ for all $e \in \mathbb{R}^+$ then $x \geq y$.

Let $x, y \in \mathbb{R}$.

Suppose $x < y$.

Let $e = \frac{y - x}{2} \in \mathbb{R}^+$. Then $x + e = x + \frac{y - x}{2} = \frac{x + y}{2} < \frac{y + y}{2} = y$, so $\exists e \in \mathbb{R}^+$, $x + e < y$.

This completes the proof by contraposition. $\square$