Basic Definitions:

A set is a collection of distinct objects, without an ordering to the objects. Two sets are equal if they share the same objects. A set is finite if the number of objects is finite and infinite otherwise.

These objects are the members or elements of the set. If an object \( x \) is a member of a set \( S \) we use the notation \( x \in S \). Otherwise we use the notation \( x \notin S \).

One basic way to specify a set is to list the objects within \( \{ \) and \( \} \), which is called set-bracket notation.

For example, \( A = \{2, 3, 5\} \) or \( B = \{\text{red, white, blue}\} \) or \( C = \{\{1, 2\}, \{1, 2, 3\}\} \) or \( \emptyset = \{} \).

This last set (with no objects) is called the empty set or the null set.

Often several elements are listed followed by ellipses (\( \ldots\)) to indicate that the pattern continues (maybe indefinitely). For instance, consider the finite set \( H = \{1, 2, 3, \ldots, 100\} \) or the infinite set \( P = \{2, 3, 5, 7, 11, \ldots\} \).

A set can also be defined in terms of a condition. Let \( D \) be a set. We can create the set \( S = \{x \in D | c_1, c_2, \ldots, c_k\} \) which is the set of elements \( x \) in \( D \) that satisfy conditions \( c_1, c_2, \ldots, c_k \).

For instance, let \( D \) be the set of people alive now and \( S = \{x \in D | x \text{ owns a vehicle.}\} \).

The cardinality of a finite set \( S \) is the number of elements in \( S \) (we’ll get into the cardinality of an infinite set later). Notation: \( |S| \).

If \( S, T \) are sets and all elements of \( S \) also belong to \( T \), we say that \( S \) is a subset of \( T \) and write \( S \subseteq T \). If \( S \neq T \) and \( S \) is a subset of \( T \) we say \( S \) is a proper subset of \( T \) and write \( S \subset T \). The empty set is a subset of every set (by default).

The notation \( S \nsubseteq T \) indicates \( S \) isn’t a subset of \( T \). Similarly, \( S \nsubseteq T \) indicates \( S \) isn’t a proper subset of \( T \).

To show that two sets are equal a standard technique is to show that they are subsets of each other. To show set \( S \) is a subset of set \( T \) just follow the following recipe: Suppose \( x \) is in \( S \). Show that it must be the case that \( x \) is in \( T \).

Often when we declare a set, in the background there is a universe: A set \( U \) in which all sets currently under consideration are subsets.
Standard definitions and notations for common sets of numbers:

The **digits** is the set \( \{0, 1, 2, \ldots, 9\} \).

The **natural numbers** is the set \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

The **integers** is the set \( \mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\} \).

Note: \( \mathbb{Z}^+ \) denotes the subset of positive integers and \( \mathbb{Z}^+ = \mathbb{N} \).

The **rationals** is the set of “fractions” made from the integers:

\[
\mathbb{Q} = \left\{ \frac{p}{q} \middle| p, q \in \mathbb{Z}, q \neq 0 \right\}.
\]

Then there’s the real numbers. Rigorous definitions of the real numbers are beyond the scope of our class. Let’s just use the following:

The **real numbers**, also known as the **reals** and denoted \( \mathbb{R} \), is the set of all signed decimal expansions:

Numbers of the form \( \pm \cdots d_2d_1d_0.d_{-1}d_{-2} \cdots = \pm \sum_{-\infty}^{\infty} d_i 10^i \) such that for integers \( i \), \( d_i \) is a digit, and all but finitely many of the digits \( d_i \) where \( i \geq 0 \) are 0.

It’s pretty easy to see that the rational numbers are a subset of the reals; just recall the division algorithm we all learn in elementary school: Any fraction \( p/q \) can be converted into a decimal expansion \( \pm \cdots d_2d_1d_0.d_{-1}d_{-2} \cdots \)

The **irrationals** are the subset of the reals that aren’t rational: \( \mathbb{I} = \{ x \in \mathbb{R} \middle| x \notin \mathbb{Q} \} \).

The **complex numbers** \( \mathbb{C} \) is the set of sums of real numbers and real numbers multiplied by the abstract number \( i = \sqrt{-1} \). In set-builder notation this is

\[
\mathbb{C} = \{ a + bi \middle| a, b \in \mathbb{R}, i^2 = -1 \}.
\]

Clearly the reals are a subset of the complex numbers: \( \mathbb{R} = \{ a + bi \in \mathbb{C} \middle| b = 0 \} \).
Let $X$ be a set. The power set of the set $X$, denoted $\mathcal{P}(X)$, is the set of all subsets of $X$ ranging from the empty set $\emptyset$ to the entire set $X$.

For example, $\mathcal{P}({a, b})\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Venn diagrams provide a visualization tool for sets: Sets are drawn as circles (sometimes in a rectangle, which depicts the universe). They can be useful as we will see.

Here is an example:
Let $S$ and $T$ be sets.

We define the **union** of $S$ and $T$ by taking all the elements that belong to $S$ or $T$ (possibly both). This is denoted $S \cup T$. For example, $\{1, 2, 3, 4, 5\} \cup \{2, 3, 5, 7\} = \{1, 2, 3, 4, 5, 7\}$.

We define the **intersection** of $S$ and $T$ by taking all the elements that belong to both $S$ and $T$. This is denoted $S \cap T$. For example, $\{1, 2, 3, 4, 5\} \cap \{2, 3, 5, 7\} = \{2, 3, 5\}$.

If $S \cap T$ is the empty set then we say these sets are **disjoint**.

We define the relative difference of $T$ in $S$ by taking all the elements that are in $S$ which are not in $T$. This is denoted $S \setminus T$ or $S - T$. For example, $\{1, 2, 3, 4, 5\} \setminus \{2, 3, 5, 7\} = \{1, 4\}$.

The **complement** of $T$, it is everything that is not in $T$. It is denoted $T'$ or $T^c$. For example, in the universe of digits, $\overline{\{2, 3, 5, 7\}} = \{0, 1, 4, 6, 8, 9\}$.

We can also take the direct product (or Cartesian product) of $S$ and $T$ which is a set, denoted $S \times T$, whose elements are ordered pairs $(s, t)$ where $s \in S$ and $t \in T$. One consequence is that if $S$ and $T$ are finite then so is $S \times T$ and $|S \times T| = |S| \cdot |T|$.

If $S = \{a, b, c\}$ and $T = \{\alpha, \beta\}$ can you list all six elements of $S \times T$?

This definition generalizes: For sets $S_1, S_2, \ldots, S_k$ we have that their direct product is $S_1 \times S_2 \times \cdots \times S_k = \{(s_1, s_2, \ldots, s_k) | s_i \in S_i \text{ for } i = 1, 2, \ldots, k\}$, and for finite sets $|S_1 \times S_2 \times \cdots \times S_k| = \prod_{i=1}^{k} |S_i|$.

Suppose $S$ and $T$ are finite sets. Let’s see how to determine the size of the union $S \cup T$.

We could add the size of $S$ and $T$. But then we have counted the intersection $S \cap T$ twice. No problem, subtract the size of $S \cap T$. Hence $|S \cup T| = |S| + |T| - |S \cap T|$; this basic counting rule is called **inclusion-exclusion**.

What does this reduce to if $S$ and $T$ are disjoint? What does the rule look like for 3 sets? $n$ sets?

How many multiples of 3 or 5 are there in the set $D = \{1, 2, 3, \ldots, 999\}$?

Let $T = \{3, 6, 9, \ldots, 999\}$, the multiples of 3 in $D$. Clearly $|T| = 333$. Let $F = \{5, 10, 15, \ldots, 995\}$, the multiples of 5 in $D$. Clearly $|F| = 200 - 1 = 199$.

Then $T \cap F = \{15, 30, 45, \ldots, 990\}$ (multiples of 15 in $D$). Clearly $|T \cap F| = 66$. Thus $|T \cup F| = 333 + 199 - 66 = 466$.

What if $D = \{111, 112, 113, \ldots, 999\}$ instead?
Suppose $I$ is a set and for any $i \in I$, $S_i$ is a set. The set $S = \{S_i| i \in I\}$ is called a family of sets indexed by $I$.

The intersection of the family $S$ is $\bigcap S = \bigcap_{i \in I} S_i = \{x| x \in S_i \text{ for all } i \in I\}$.

The union of the family $S$ is $\bigcup S = \bigcup_{i \in I} S_i = \{x| x \in S_i \text{ for some } i \in I\}$.

Consider the family $S = \left\{ \left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right) \bigg| n \in \mathbb{N} \right\}$ of open intervals in $\mathbb{R}$.

Then
\[
\bigcap S = \bigcap_{n \in \mathbb{N}} \left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right) = [0, 1],
\]
since every number in the closed interval $[0, 1]$ is in $\left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right)$ for any $n \in \mathbb{N}$ and for any real not in $[0, 1]$ there is a sufficiently large $n \in \mathbb{N}$ such that it is not in $\left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right)$.

Furthermore,
\[
\bigcup S = \bigcup_{n \in \mathbb{N}} \left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right) = (-1, 2),
\]
since for $n = 1$ we have $\left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right) = (-1, 2)$ and for $n \in \mathbb{N}$, $n > 1$, we have $\left( 0 - \frac{1}{n}, 1 + \frac{1}{n} \right) \subseteq (-1, 2)$.

A partition of a set $S$ is a family $\mathcal{C} = \{P_i| i \in I\}$ of nonempty, pairwise disjoint sets indexed by some set $I$ such that $\bigcup_{i \in I} P_i = S$.

For instance, $\mathcal{C} = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ is a partition of $S = \{1, 2, 3, 4, 5\}$.

Basic counting rule: If $\mathcal{C} = \{P_i| i \in I\}$ is a partition of a finite set $S$. Then $|S| = \sum_{i \in I} |P_i|$.
The problem with classic set theory: When we define a set using a condition, does such a set always exist?

Consider the following set:

Let $R$ be the set of all sets that are not members of themselves. Symbolically

$$R = \{ S \in U \mid S \notin S \},$$

where $U$ is the universe of all sets.

If $R$ is not an element of itself, then it must contain itself. If it contains itself, then it is not an element of itself. This is a fun contradiction known as “Russell’s Set Theory Paradox.”

Therefore, no such set $R$ exists. Such a set is an example of an ill-defined set. The remedy? We exclude sets that are ill-defined from our universe, rather requiring that all sets are well-defined (actually exist).

DeMorgan’s Laws (verify by using Venn diagrams):

1. \[ \overline{S \cup T} = \overline{S} \cap \overline{T}. \]
2. \[ \overline{S \cap T} = \overline{S} \cup \overline{T}. \]
1.4-1.6 Basic Logic

A sentence that is true of false, but not both, is a proposition. It is typical to use labels for propositions.

“It is snowing.” is a proposition, while “this statement is false” is not. (Why?)

For example, $p$ may stand for the proposition $2 + 2 = 1$ (false for $\mathbb{Z}$, but true for $\mathbb{Z}$ modulo 3 (more on this later).)

Propositions may be combined by using connectives such as and/or.

Notation: $p \lor q$ denotes “$p$ or $q$”. This is called a disjunction. The only case in which this is false is when both are false.

Notation: $p \land q$ denotes “$p$ and $q$”. This is called a conjunction. The only case in which this is true is when both are true.

Propositions may be negated to the opposite truth value.

Notation: $\neg p$ denotes “not $p$”.

Truth tables give the truth of a compound statement based upon all possible truth values of the inputs. Example: Let’s write a truth table for $\neg p \lor q$.

Write the truth tables for the following compound statements:

(1) $\neg (p \lor (q \land r))$.

(2) $\neg (p \land q) \lor (\neg q \lor r)$. 
Let $p, q$ be propositions. A statement of the form “If $p$ then $q$” is called an implication or a conditional proposition. We will use the notation $p \Rightarrow q$.

$p$ is called the antecedent or hypothesis and $q$ is called the consequent or conclusion.

The only situation in which $p \Rightarrow q$ is false is when $p$ is true and $q$ is false. In all other situations, $p \Rightarrow q$ is true.

Assume $p$ and $q$ is true, while $r$ is false. Determine the truth value of $p \Rightarrow (q \Rightarrow r)$.

When $p \Rightarrow q$ is true on account of $p$ being false, the conditional statement is said to be vacuously true or true by default.

Other ways to state if $p$ then $q$:

1. $q$ if $p$
2. $p$ only if $q$.
3. $p$ implies $q$.
4. $q$ whenever $p$.

For example, the statement it is raining only if it is cloudy is equivalent to if it is raining then it is cloudy.

**Careful:** If it is cloudy then it is raining is not an equivalent statement.

Given $p \Rightarrow q$, the reverse conditional statement, $q \Rightarrow p$, is called the converse and is not logically equivalent. The conditional statement made with the negation of each part of a conditional statement, $\neg p \Rightarrow \neg q$, is called the inverse and is also not logically equivalent. The conditional statement, $\neg q \Rightarrow \neg p$ is called the contrapositive and is logically equivalent (verify by a truth table).

Contrapositive example: If it is not cloudy then it is not raining.

Write the conditional proposition, “if Jamie is a husband then Jamie got married” symbolically. Then write its converse, inverse and contrapositive symbolically and in words.

$p$: Jamie is a husband.
$q$: Jamie got married.

Original: $p \Rightarrow q$.
Converse: $q \Rightarrow p$. (If Jamie got married then Jamie is a husband.)
Inverse: $\neg p \Rightarrow \neg q$. (If Jamie is not a husband then Jamie didn’t get married.)
Contrapositive: $\neg q \Rightarrow \neg p$. (If Jamie didn’t get married then Jamie is not a husband.)
A necessary condition is a condition that must be satisfied for a particular outcome to be achieved. The conclusion of a conditional statement expresses a necessary condition. A sufficient condition is a condition that suffices to guarantee a particular outcome. The hypothesis of a conditional statement expresses a sufficient condition.

Consider the proposition “if the real number $x$ is larger than 0 then $x^2$ is larger than zero.” A necessary condition of $x > 0$ is $x^2 > 0$. A sufficient condition for $x^2 > 0$ is $x > 0$.

A “$p$ if and only if $q$” statement is a biconditional proposition. Sometimes, people abbreviate with “p iff q.” Often in mathematical definitions, we use if, but mean iff.

Notation: $p \iff q$. It’s true exactly when $p$ and $q$ have the same truth values.

In the case of $p \iff q$, you could also say, $p$ is a necessary and sufficient condition for $q$.

The truth table is pretty nice. $p \iff q$ is true if and only if the truth values of $p$ and $q$ agree.

**Definition:** Two propositions $P$ and $Q$, made up of propositions $p_1, ..., p_n$, are logically equivalent if $P$ and $Q$ have the same truth values for all possible combinations of truth values for propositions $p_1, ..., p_n$.

Notation: $P \equiv Q$.

For instance we have the disjunctive equivalence of the implication (it is left to you to verify):

$$(p \Rightarrow q) \equiv \neg p \lor q.$$
DeMorgan’s Laws (verify by using truth tables):

\[
\begin{align*}
(1) \quad \neg(p \lor q) & \equiv \neg p \land \neg q. \\
(2) \quad \neg(p \land q) & \equiv \neg p \lor \neg q.
\end{align*}
\]

How would we negate \( p \to q \)?

Begin with the disjunctive equivalence,

\[(p \Rightarrow q) \equiv (\neg p \lor q).\]

So, \( \neg(p \Rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q. \)
Let $P(x)$ be a statement involving the variable $x$ and let $D$ be a set. We call $P$ a propositional function over the domain $D$ if for each $x \in D$, $P(x)$ is a proposition.

Example:

The city $X$ has more than 1 million residents. The domain could be any set of cities.

Let $P(x)$ be a prepositional function with domain $D$.

The statement “for all $x \in D$, $P(x)$” denoted
\[
\forall x \in D, P(x),
\]
is called a universally quantified statement. The symbol $\forall$ (universal quantifier) means “for all” (interchangeable with “for any” or “for every” or “for each”).

When $D$ is clear the universally quantified statement can be written as $\forall x, P(x)$.

The statement $\forall x, P(x)$ is true if $P(x)$ is true for every $x$ in the domain $D$.

That means that if $P(x)$ is false for at least one $x$ in the domain $D$ then the statement $\forall x, P(x)$ is false.

To show the statement $\forall x, P(x)$ is true you must show it is true for every choice of $x$ from $D$.

To show $\forall x, P(x)$ is false you must produce just one counterexample: choice of $x$ for which $P(x)$ is false.

Consider the following statement:

\[
\forall x \in \mathbb{R}, \frac{1}{1 + x^2} \leq 1.
\]

This is true: Let $x$ be an arbitrary real number. Then $x^2 \geq 0$. By adding one to both sides, $1 + x^2 \geq 1$. By taking the reciprocal of both sides (larger positive denominator means smaller fraction)
\[
\frac{1}{1 + x^2} \leq 1.
\]
Consider the following minor adjustment in the previous statement:

\[ \forall x \in \mathbb{R}, \frac{1}{1 + x^2} < 1. \]

This is false: Having \( x = 0 \) gives us a counterexample as then \( \frac{1}{1 + x^2} = 1 \).

Let \( P(x) \) be a prepositional function with domain \( D \).

The statement “there exists \( x \in D \) such that \( P(x) \)” denoted

\[ \exists x \in D, P(x), \]

is called an existentially quantified statement. The symbol \( \exists \) (existential quantifier) means “there exists” (interchangeable with “for some”).

When \( D \) is clear the universally quantified statement can be written as \( \exists x, P(x) \).

The statement \( \exists x, P(x) \) is true if \( P(x) \) is true for at least one \( x \) in the domain \( D \).

That means that if \( P(x) \) is false for every \( x \) in the domain \( D \) then the statement \( \exists x, P(x) \) is false.

To show the statement \( \exists x, P(x) \) is true you must show it is true for a particular \( x \) from \( D \).

To show \( \exists x, P(x) \) is false you must show \( P(x) \) is false for every \( x \) in \( D \).

Consider the following statement:

\[ \exists x \in \mathbb{R}, x^2 < x. \]

This statement is true: For \( x = 0.5 \) we have \( x^2 < x \).

\[ \exists x \in \mathbb{R}, x > 1 \text{ and } x^2 < x. \]

This statement is false: Let \( x \) be a real number. Two cases: Either \( x > 1 \) or not. If not then the conjunction “\( x > 1 \) and \( x^2 < x \)” is false. Suppose \( x > 1 \). Then \( x - 1 > 0 \). Then \( x(x - 1) > 0 \) (since the product of two positives is positive). So \( x^2 - x > 0 \) implying \( x^2 > x \). So the conjunction “\( x > 1 \) and \( x^2 < x \)” is false in either case.
DeMorgan’s laws for negating quantified statements:

(1) \( \neg (\forall x, \ P(x)) \equiv \exists x, \ \neg P(x). \)

(2) \( \neg (\exists x, \ P(x)) \equiv \forall x, \ \neg P(x). \)

These follow directly from the definition of the quantified statements.

Let \( S \) be a subset of \( \mathbb{R} \). We say \( S \) has the Archimedean property if and only if
\( \forall a, b \in S, \ \exists n \in \mathbb{N} \) such that \( na > b \).

Let’s write the negation:
\( \neg (\forall a, b \in S, \ \exists n \in \mathbb{N} \text{ such that } na > b) \equiv \exists a, b \in S, \text{ such that } \forall n \in \mathbb{N}, \ na \leq b. \)

For instance, \( \mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\} \) has the Archimedean property, but \( \mathbb{Z} \) doesn’t

We can nest quantifiers within quantifiers (any number of nestings). For instance consider the statement:

\[ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \ \frac{y}{x} \in \mathbb{Z}. \]

This statement is true: We can let \( x = 1 \). Then for an arbitrary \( y \in \mathbb{Z} \) we have \( \frac{y}{1} = y \), which is in \( \mathbb{Z} \).

Would it still be true if \( \frac{y}{x} \) was \( \frac{x}{y} \)?

Order matters! Consider the following two statements:

(1) \( \forall x \in \mathbb{R}, \ \exists y \in \mathbb{N}, \ |x| < y. \)

(2) \( \exists y \in \mathbb{N}, \forall x \in \mathbb{R}, \ |x| < y. \)

The first one is true, but the second one is false. Can you justify this?

Careful: Switching the order doesn’t always result in the opposite truth value (see our first example preceding these two).
Let $f(x)$ be a single variable function, and let $a, L$ be fixed real numbers. Consider the following statement (motivated by limits from calculus):

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all reals $x$, $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

This involves nesting three quantifiers! It is the technical definition of

$$\lim_{x \to a} f(x) = L.$$ 

Can you negate this definition?