MULTIPLICATION PRINCIPLE:

If there \( m \) ways to do something and \( n \) ways to do another thing then there are \( mn \) ways to do both.

In the language of set theory: Let \( S, T \) be sets with \( m, n \) elements respectively. If \( S \) is a set of possible outcomes for a first task and \( T \) is the set of possible outcomes for a second task then the set of outcomes for completing both tasks is \( S \times T \) and \(|S \times T| = |S||T|\).

This generalizes to \( n \) tasks: If \( m_i \) counts the number of ways to complete task \( i \) for \( i = 1, 2, ..., n \) then \( \prod_{i=1}^{n} m_i \) counts the ways to do all the tasks.

Example: How many positive divisors does 10,000 = \( (10)^4 = 2^45^4 \) have?

Each positive divisor corresponds to \( 2^i5^j \) where \( i \) and \( j \) are chosen from \( \{0, 1, 2, 3, 4\} \). So by the multiplication principle there are \((5)(5) = 25\) positive divisors of 10,000.

Theorem: A set of \( n \) objects has \( 2^n \) subsets.

Proof: A subset of an \( n \)-element set is equivalent to a unique labeling of each of the \( n \) elements with either a 0 or a 1 (0 means “not in the subset” and 1 means “in the subset”). So the number of subsets is equal to the number of ways to label each of the \( n \) elements with either a 0 or a 1. That’s \( 2^n \) by the multiplication principle \( \square \)

Let \( k, n \) be non-negative integers. A \( k \)-element permutation of \( n \) objects is an ordered list of \( k \) of the \( n \) objects. The number of \( k \)-element permutations of \( n \) objects is denoted \( P(n, k) \).

By the multiplication principle,

\[
P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}, \text{ where } 0! = 1 \text{ and for } m \in \mathbb{N} \text{ we have } m! = \prod_{i=1}^{m} i.
\]

Example: A traditional starting lineup in basketball consists of a point guard, a shooting guard, a power forward, a small forward, and a center. How many traditional lineups can be formed from a 12 person roster if any player can play any position?

It’s an ordered list of 5 of the 12; there are \( P(12, 5) = (12)(11)(10)(9)(8) = 95,040 \) lineups.
Let $k, n$ be non-negative integers and $k \leq n$. A $k$-element combination (aka selection) from $n$ objects is an unordered collection of $k$ of the $n$ objects (equivalently a $k$-element subset of the set of $n$ objects). So, the number of $k$-element subsets of an $n$-element set is equal to the number of $k$-element combinations (aka selections) taken from $n$ objects. This number is denoted $C(n, k)$ or $\binom{n}{k}$. It’s read “$n$ choose $k$.”

I prefer the $C(n, k)$ notation most of the time.

Let’s derive a formula for $C(n, k)$. A $k$-element permutation of $n$ objects can be uniquely determined by the following two step process: Select a $k$-element combination from the $n$ objects, and then put these $k$ objects in a particular order. The number of $k$-element combinations is $C(n, k)$ and the number of ways to put the $k$ objects selected into an order is $k(k-1) \cdots \cdot (2)(1) = k!$, so by the multiplication principle,

$$P(n, k) = C(n, k) \cdot k!.$$ 

Hence $C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}$.

Example: Jack has a coupon for a 3-topping large pizza from a local pizzeria. How many choices does Jack have for his 3-topping large pizza, given that the pizzeria has 10 total topping choices?

The answer is 10 choose 3 (which is $C(10, 3) = \frac{(10)(9)(8)}{(3)(2)} = 120$ by the formula above).

How many 4 digit strings have exactly 3 ones? Note: A digit string of length 4 is a 4-element permutation of the digits.

There are $C(4, 3) = 4$ locations in which the ones can occur, and 9 choices for the digit not equal to 1. So by the multiplication principle we get $(4)(9) = 36$ such strings.

Theorem: Suppose $k, n$ are non-negative integers and $k \leq n$. Then $C(n, k) = C(n, n - k)$.

Proof: Choosing $k$ elements from $n$ objects is equivalent to choosing $n - k$ elements to leave behind. Therefore the number of selections of $k$ elements from $n$ is the same as the number of selections of $n - k$ objects from $n$. □
**Theorem**: For any integer $n \geq 2$, and for any positive integer $k < n$ we have
\[ C(n, k) = C(n - 1, k - 1) + C(n - 1, k). \]

**Proof**: Let $X$ be a set of $n \geq 2$ objects. Let $S$ be the set of $k$-element combinations (hence subsets) of the $n$ objects of $X$. Thus $|S| = C(n, k)$. Let $x \in X$. We can partition $S$ into two subsets $A$ and $B$, where $A$ are the $k$-element combinations that contain $x$ and $B$ are the $k$-element combinations that do not contain $x$. Clearly $A$ and $B$ are disjoint and their union is $S$. So $|S| = |A| + |B|$. The cardinality of $A$ is $C(n - 1, k - 1)$ since to form a $k$-element combination already containing $x$, $k - 1$ more elements must be selected from the $n - 1$ elements left in $X \setminus \{x\}$. The cardinality of $B$ is $C(n - 1, k)$ since to form a $k$-element combinations not containing $x$, all $k$ elements must be selected from the $n - 1$ elements left in $X \setminus \{x\}$. That completes the proof \(\square\)

**Pascal’s triangle**:

Begin with a triangle of 3 ones in two rows. The rows below extend the triangle and are generated by putting 1 on each end and in other locations adding the two entries directly above as follows:

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
\vdots & & & & \ddots & \\
\end{array}
\]

We call the top row the $O$th row and it’s $C(0, 0) = 1$. The next row is the 1st row and it’s $C(1, 0) C(1, 1)$. Now let $n \geq 2$ be an integer. The fact that $C(n, 0) = C(n, n) = 1$ and the previous theorem tells us that the $n$th row of the triangle is:

\[ C(n, 0) C(n, 1) C(n, 2) \ldots C(n, n - 2) C(n, n - 1) C(n, n) \]
Binomial Theorem: If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ then

$$(x + y)^n = \sum_{k=0}^{n} C(n, k) \cdot x^{n-k} y^k.$$ 

Proof: We shall return to the method of induction. Suppose $n = 1$. Then $(x + y)^n = x + y$ and $\sum_{k=0}^{n} C(n, k) \cdot x^{n-k} y^k = C(1, 0)xy^0 + C(1, 1)x^0y = x + y$. This establishes the base case.

Let $n \in \mathbb{N}$. Assume $(x + y)^n = \sum_{k=0}^{n} C(n, k) \cdot x^{n-k} y^k$. Then we multiply both sides by $(x + y)$ and simplify:

$$(x + y)^n (x + y) = \left( \sum_{k=0}^{n} C(n, k) \cdot x^{n-k} y^k \right) \cdot (x + y),$$

$$(x + y)^{n+1} = \sum_{k=0}^{n} C(n, k) \left( x^{n-k+1} y^k + x^{n-k} y^{k+1} \right),$$

$$(x + y)^{n+1} = x^{n+1} + x^n y + \left( \sum_{k=1}^{n-1} C(n, k) x^{n-k+1} y^k + \sum_{k=1}^{n-1} C(n, k) x^{n-k} y^{k+1} \right) + xy^n + y^{n+1},$$

$$(x + y)^{n+1} = x^{n+1} + \left( \sum_{i=1}^{n} C(n, i) x^{n-i+1} y^i + \sum_{j=0}^{n-1} C(n, j) x^{n-j} y^{j+1} \right) + y^{n+1},$$

$$(x + y)^{n+1} = x^{n+1} + \left( \sum_{i=1}^{n} C(n, i) x^{n-i+1} y^i + \sum_{i=1}^{n} C(n, i-1) x^{n-i+1} y^i \right) + y^{n+1},$$

$$(x + y)^{n+1} = x^{n+1} + \left( \sum_{i=1}^{n} (C(n, i) + C(n, i-1)) x^{n+1-i} y^i \right) + y^{n+1},$$

$$(x + y)^{n+1} = x^{n+1} + \left( \sum_{i=1}^{n} C(n + 1, i) x^{n+1-i} y^i \right) + y^{n+1},$$

$$(x + y)^{n+1} = \sum_{i=0}^{n+1} C(n + 1, i) x^{n+1-i} y^i,$$

The second-to-last step uses that $C(n + 1, i) = C(n, i) + C(n, i - 1)$ for $i = 1, 2, ..., n$ by a previous theorem \(\square\)
Let’s apply the binomial theorem to find the coefficient of \(a^4b^3\) in \((2a - b)^7\).

Using \(x = 2a\), \(y = -b\) and selecting the term of the sum corresponding to \(k = 3\) we get \(\binom{7}{3}(2a)^4(-b)^3 = 35(-16a^4b^3) = -560a^4b^3\). So the coefficient is \(-560\).

**Multinomial Theorem:** Let \(m, n \in \mathbb{N}\) and \(m \geq 2\).

Then \((x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + k_2 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{1 \leq t \leq m} x_t^{k_t}\),

where \(\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{(k_1!)(k_2!)(k_3!)\cdots(k_m!)}\) is a multinomial coefficient. The sum is taken over all combinations of nonnegative integer indices \(k_1 \text{ through } k_m\) that sum to \(n\).

**Proof:** Let \(n \in \mathbb{N}\). We shall use induction on \(m\) and the binomial theorem. When \(m = 2\) the theorem holds as it’s the Binomial theorem. Throughout assume that \(k\) and \(k_j\) for any \(j\), all represent non-negative integers.

Let \(m \geq 2\) be an integer. Assume that \((x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + k_2 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{1 \leq t \leq m} x_t^{k_t}\).

Then \((x_1 + x_2 + \cdots + x_m + x_{m+1})^n = (x_1 + x_2 + \cdots + x_{m-1} + (x_m + x_{m+1}))^n = \sum_{k_1 + k_2 + \cdots + k_{m-1} + k = n} \binom{n}{k_1, k_2, \ldots, k_{m-1}, k} \prod_{1 \leq t \leq m-1} x_t^{k_t} (x_m + x_{m+1})^k = \sum_{k_1 + k_2 + \cdots + k_{m-1} + k = n} \binom{n}{k_1, k_2, \ldots, k_{m-1}, k} \prod_{1 \leq t \leq m-1} x_t^{k_t} \sum_{k_m + k_{m+1} = k} \binom{k}{k_m} x_m^{k_m} x_{m+1}^{k_{m+1}} = \sum_{k_1 + k_2 + \cdots + k_m + k_{m+1} = n} \binom{n}{k_1, k_2, \ldots, k_m, k_{m+1}} \prod_{1 \leq t \leq m+1} x_t^{k_t}\)

since \(\binom{n}{k_1, k_2, \ldots, k_m, k} \binom{k}{k_m} = \frac{n!}{(k_1!)(k_2!)(k_3!)\cdots(k_m!) (k_m!)(k_{m+1}!)} = \frac{n!}{(k_1!)(k_2!)(k_3!)\cdots(k_{m-1}!)(k_m!)(k_{m+1}!)} \binom{n}{k_1, k_2, \ldots, k_m, k_{m+1}}\).

That completes the proof by induction \(\square\)

What’s the coefficient of \(x^3y^2z^2\) in \((x + y + z)^7\)?

It’s \(\binom{7}{3, 2, 2} = \frac{7!}{(3!)(2!)(2!)} = (7)(6)(5) = 210\).
Theorem: Suppose a sequence of length \( n \) has \( n_i \) identical terms of type \( i \), for \( 1 \leq i \leq k \). Then the number of distinct orderings of this sequence is the multinomial coefficient 
\[
\binom{n}{n_1, n_2, \ldots, n_k}.
\]

Proof: Let \( a_1, a_2, \ldots, a_n \) be a sequence of length \( n \) with \( n_i \) identical terms of type \( i \), for \( 1 \leq i \leq k \). In particular, \( n_1 + n_2 + \cdots + n_k = n \).

Consider a sequence of \( n \) empty slots \( \square \square \square \cdots \square \). There are \( \binom{n}{n_1} \) ways to insert the \( n_1 \) terms of type 1 into these slots. Now there only remain \( n - n_1 \) empty slots. There are \( \binom{n - n_1}{n_2} \) ways to insert \( n_2 \) terms of type 2 into these remaining slots. Continuing like way we get \( \binom{n - n_1 - n_2 - \cdots - n_{\ell-1}}{n_\ell} \) ways to insert the \( n_\ell \) terms of type \( \ell \) into the remaining slots for \( \ell = 3, 4, \ldots, k \). Then by the multiplication principle, the number of distinct orderings of \( a_1, a_2, \ldots, a_n \) is
\[
\binom{n}{n_1} \binom{n - n_1}{n_2} \binom{n - n_1 - n_2}{n_3} \cdots \binom{n - n_1 - n_2 - \cdots - n_{k-1}}{n_k} =
\]
\[
= \frac{n!}{n_1!(n - n_1)!} \cdot \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \cdot \frac{(n - n_1 - n_2)!}{n_3!(n - n_1 - n_2 - n_3)!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}
\]
\( \Box \)

Example: How many distinct alphabet strings can be made with the letters of the word \textit{BLUEBERRIES}?

There are 3 E’s, 2 B’s, 2 R’s, and 1 each of I,L,S,U. So there are 
\[
\binom{11}{3, 2, 2, 1, 1, 1, 1} = \frac{11!}{3!(2!)^2} = \frac{11!}{4!} = (11)(10)(9)(8)(7)(6)(5) = 1,663,200 \text{ such strings.}
\]
Now let’s consider selections where order is not taken into account, but repetition is allowed.

Let’s start with a simple example. Suppose we have a bag that contains 3 indistinguishable red balls, 3 indistinguishable white balls, and 3 indistinguishable blue balls. How many distinct drawings (ignoring order) of 3 balls are possible? This one is easy enough that we can find the selections by brute force (meaning we list all the selections):

We will use “R,” “W,” and “B” to denote a red, white, and blue ball respectively. Since order doesn’t matter let’s adopt the convention of writing R before W and W before B. Here are all the possibilities:

Involving only one color: RRR, WWW, BBB. Involving exactly two colors: RRR, RBB, RW W, WW B, RBB, W BB. Involving all three colors: RW B.

So there are 10 distinct drawings. Incidentally, what’s \( C(5, 2) \)?

**Theorem:** If \( X \) is a set containing \( t \) elements, then the number of unordered \( k \)-term selections from \( X \), where repetitions are allowed, is \( C(k + t - 1, t - 1) = C(k + t - 1, k) \).

**Proof:** Let \( X = \{a_1, a_2, ..., a_t\} \). We will show that every unordered \( k \)-term selection from \( X \), where repetitions are allowed, corresponds to a diagram where \( t - 1 \) of the \( k + (t - 1) \) slots

\[
\square \square \square \cdots \square
\]

are filled with the “bar” symbol \( | \) and vice versa (every diagram corresponds to a selection). After that it will be simple to complete the proof.

Suppose we have an unordered \( k \)-term selection from \( X \) where repetitions are allowed. Let \( n_i \) count the number of times \( a_i \) appears in the selection for \( i = 1, 2, ..., t \). Clearly \( n_1 + n_2 + \cdots + n_t = k \). Let’s build a diagram by starting with \( n_1 \) slots, followed by a bar, followed by \( n_2 \) slots, followed by a bar, ..., followed by \( n_{t-1} \) slots, followed by a bar, followed by \( n_t \) slots. What we get is a diagram where \( t - 1 \) of the \( k + (t - 1) \) slots

\[
\square \square \square \cdots \square
\]

are filled with the bar symbol.

Now suppose we have such a diagram. Let’s build the selection corresponding to the diagram. The number of slots to the left of the first bar (maybe 0 slots) is the number of times \( a_1 \) appears in the selection. For \( i = 2, 3, ..., t - 1 \), we have that the number of slots between the \( (i - 1) \)st bar and the \( i \)th bar (maybe 0 slots) is the number of times \( a_i \) appears in the selection. The number of slots to the right of the last bar (maybe 0 slots) is the number of times \( a_t \) appears in the selection. Since there are a total of \( k \) slots amongst the \( t - 1 \) bars, we end up with an unordered \( k \)-term selection from \( X \) where repetitions are allowed.

So counting the number of unordered \( k \)-term selections from \( X \), where repetitions are allowed, is equivalent to counting how many diagrams there are where \( t - 1 \) of \( k + (t - 1) \) slots are filled with the bars. So we get \( C(k + t - 1, t - 1) = C(k + t - 1, k) \) (same as choosing \( k \) slots to not be bars) \( \square \)
Example: Suppose a donut shop is having a special on a dozen donuts where customers must choose their dozen from 5 different types of donuts. How many distinct dozens can be made from the 5 types?

In this example customers select \( k = 12 \) donuts from a set of \( t = 5 \) types (repetition allowed). So by the previous theorem, there are

\[
C(12 + 4, 4) = C(16, 4) = \frac{(16)(15)(14)(13)}{4!} = 1,820
\]
distinct dozens that can be ordered.

Now let’s consider how to find how many non-negative integer solutions there are to the equation

\[x_1 + x_2 + \cdots + x_t = n.\]

We can “bars and slots” idea since a solution to the equation corresponds to a diagram where \( t - 1 \) of \( n + (t - 1) \) slots are filled with bars. Let’s see why.

Consider such a diagram. Let \( x_1 \) be the number of slots to the left of the first bar. For \( i = 2, 3, ..., t - 1 \), let \( x_i \) be the number of slots between the \((i - 1)\)st bar the the \(i\)th bar. Let \( x_t \) be the number of slots to the right of the last bar. Then we have a solution to the equation.

It is also easily seen that a solution can be coded as such a diagram.

Therefore, by the theorem, the number of non-negative integer solutions to the equation is

\[C(n + t - 1, t - 1).\]

Example: Consider the equation \( x + y + z = 10 \). There are \( C(12, 2) = (12)(11)/2 = 66 \) non-negative integer solutions.

(ELFY) Solve the problem if we assume the integers have to be positive. Hint: Substitute \( x'_i = x_i - 1 \) into the equation and count non-negative integer solutions to the resulting equation...

Consider the following example:

15 identical dolls are to be distributed into bins marked 1, 2, 3, 4. In how many ways can this happen?

This is the same as asking how many non-negative integer solutions are there to the equation \( x_1 + x_2 + x_3 + x_4 = 15 \). So the solution is \( C(18, 3) = 816 \).
Recursion is helpful for solving some counting problems. We will begin with an example:

How many digit strings of length 4 do not contain the pattern 00?

Let $S_n$ be the number of digit strings of length $n \in \mathbb{N}$ that do not contain the pattern 00. Clearly $S_1 = 10$ and $S_2 = 99$.

Let $n \geq 3$. We partition digit string of length $n$ that do not contain the pattern 00 into two sets, those that begin with a 0 and those that don’t.

If the first digit isn’t a 0 then there are 9 choices for that digit followed by $S_{n-1}$ ways to fill the rest of the string.

If the first digit is a 0 then there are 9 choices for the second digit followed by $S_{n-2}$ ways to fill the rest of the string.

So we get $S_n = 9S_{n-1} + 9S_{n-2} = 9(S_{n-1} + S_{n-2})$. So $S_3 = 9(S_2 + S_1) = 9(99 + 10) = 981$ and $S_4 = 9(S_3 + S_2) = 9(981 + 99) = 9,720$.

So there are 9,720 such strings.

What if we want a closed formula for $n$ digit strings? For that purpose we prove the following:

**Theorem**: Suppose the recursion $a_n = c_1a_{n-1} + c_2a_{n-2}$ holds or $n \geq 3$ and $a_1, a_2$ are given. If $r$ is a root of $t^2 - c_1t - c_2 = 0$, then the sequence $a_n = r^n$ solves the recursion. If the polynomial above has roots $r_1 \neq r_2$, then there exists constants $b, c$ such that

$$a_n = br_1^n + cr_2^n.$$

**proof**: If $r$ is a root of $t^2 - c_1t - c_2 = 0$ then $r^2 = c_1r + c_2$. Then

$$c_1r^{n-1} + c_2r^{n-2} = r^{n-2}(c_1r + c_2) = r^{n-2}r^2 = r^n.$$

So $r_1^n$ solves the recursion.

It’s easy to see that any linear combination of $r_1$ and $r_2$ will satisfy the recursion.

Since $r_1 \neq r_2$ there exist constants $b, c$ such that $br_1 + cr_2 = a_1$ and $br_1^2 + cr_2^2 = a_2$. Thus

$$a_n = br_1^n + cr_2^n \square$$
So let’s return the recursion \( S_n = 9(S_{n-1} + S_{n-2}) \) with \( S_1 = 10 \) and \( S_2 = 99 \).

Let’s find the roots of \( t^2 - 9t - 9 \). By the quadratic formula the roots are \( t = \frac{9 \pm 3\sqrt{13}}{2} \).

We can find \( b, c \) such that
\[
b \left( \frac{9 + 3\sqrt{13}}{2} \right)^2 + c \left( \frac{9 - 3\sqrt{13}}{2} \right)^2 = 10 \text{ and}
\]
\[
b \left( \frac{9 + 3\sqrt{13}}{2} \right)^2 + c \left( \frac{9 - 3\sqrt{13}}{2} \right)^2 = 99.
\]

(ELFY) Solve the system above to show that \( b = \frac{2(65 + 18\sqrt{13})}{39(3 + \sqrt{13})} \) and \( c = \frac{-11 + 3\sqrt{13}}{6\sqrt{13}} \).

Hence \( S_n = \frac{2(65 + 18\sqrt{13})}{39(3 + \sqrt{13})} \left( \frac{9 + 3\sqrt{13}}{2} \right)^n + \frac{-11 + 3\sqrt{13}}{6\sqrt{13}} \left( \frac{9 - 3\sqrt{13}}{2} \right)^n \) or \( n = 1, 2, ... \)

Plugging in \( n = 4 \) yields the same answer, 9,720. The advantage? We can plug in \( n = 10 \) for example (better than going through the recursion one step at a time).

Fun with the counting numbers:

Let’s denote by \( d_n \) the number of ways to write \( n \in \mathbb{N} \) as a sum of distinct positive integers.

We can quickly determine some of the first few values of this sequence:
\( d_1 = 1 \) as 1 is the only way to do it, \( d_2 = 1 \) as 2 is it, \( d_3 = 2 \) as we can either write 3 or \( 2 + 1 \), \( d_4 = 2 \) as we have 4 and \( 3 + 1 \), \( d_5 = 3 \) as we have 5, \( 4 + 1 \), and \( 3 + 2 \), \( d_6 = 4 \) as we have 6, \( 5 + 1 \), \( 4 + 2 \), and \( 3 + 2 + 1 \).

(ELFY) Explain why \( d_7 = 5 \).

Let’s denote by \( o_n \) the number of ways to write \( n \in \mathbb{N} \) as a sum of odd positive integers.

We can quickly determine some of the first few values of this sequence:
\( o_1 = 1 \) as it, \( o_2 = 1 \) as \( 1 + 1 \) is it, \( o_3 = 2 \) as we have 3 and \( 1 + 1 + 1 \), \( o_4 = 2 \) as we have \( 3 + 1 \) and \( 1 + 1 + 1 + 1 \), \( o_5 = 3 \) as we have 5, \( 3 + 1 + 1, 1 + 1 + 1 + 1 + 1 \), \( o_6 = 4 \) as we have \( 5 + 1, 3 + 3, 3 + 1 + 1 + 1, \) and of course \( 1 + 1 + 1 + 1 + 1 + 1 \).

(ELFY) Explain why \( o_7 = 5 \).
In 1740 Euler proved the following in a very clever way:

**Theorem:** For all \( n \in \mathbb{N} \), \( d_n = o_n \).

**Proof:** Let’s find a pattern in expanding the infinite product \((1+x)(1+x^2)(1+x^3)(1+x^4)\cdots(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots + d_nx^n + \cdots\), since for any sum of distinct integers adding to \( n \) there is a path of choices of factors from the first factor continuing indefinitely (eventually always choosing 1) that produces a contribution of \( x^n \) in the expanded infinite product.

Recall the geometric sum rule:

\[
\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.
\]

Now consider expansion of the infinite product shown below:

\[
\left( \frac{1}{1-x} \right) \left( \frac{1}{1-x^3} \right) \left( \frac{1}{1-x^5} \right) \left( \frac{1}{1-x^7} \right) \cdots =
(1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^7 + x^{14} + \cdots) \cdots =
\]

\[
(1 + x + x^{1+1} + \cdots)(1 + x^3 + x^{3+3} + \cdots)(1 + x^5 + x^{5+5} + \cdots)(1 + x^7 + x^{7+7} + \cdots) \cdots =
1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots + o_nx^n + \cdots,
\]

since for any sum of odd integers adding to \( n \) there is a path of choices of factors from the first factor continuing indefinitely (eventually always choosing 1) that produces a contribution of \( x^n \) in the expanded infinite product.

To elaborate, the path consists of picking \( x \) number of 1’s in the sum from the first factor, and \( (x^3) \) number of 3’s in the sum from the second factor, and so on...

It only remains to show that these two infinite products are actually equal:

\[
\left( \frac{1}{1-x} \right) \left( \frac{1}{1-x^3} \right) \left( \frac{1}{1-x^5} \right) \left( \frac{1}{1-x^7} \right) \cdots =
\]

\[
= \left( \frac{1}{1-x} \right) \left( \frac{1-x^2}{1-x^2} \right) \left( \frac{1-x^4}{1-x^4} \right) \left( \frac{1-x^6}{1-x^6} \right) \left( \frac{1}{1-x^7} \right) \cdots =
\]

\[
= \left( \frac{1}{1-x} \right) \left( \frac{(1-x)(1+x)}{1-x^2} \right) \left( \frac{1-x^2}{1-x^4} \right) \left( \frac{1-x^4}{1-x^5} \right) \left( \frac{1-x^6}{1-x^6} \right) \left( \frac{1}{1-x^7} \right) \cdots =
(1 + x)(1 + x^2)(1 + x^3) \cdots \Box
\]
A standard method for proving that an equation involving binomial coefficients is true is to show that both sides count the same thing. Such an argument is called a combinatorial argument.

We have examples of this already, such as the proof of \( C(n, k) = C(n, n - k) \) for non-negative integers \( k, n \) where \( k \leq n \), or the proof of \( C(n, k) = C(n - 1, k - 1) + C(n - 1, k) \) for positive integers \( k, n \) where \( k < n \).

Here are more examples:

**Theorem (Vandermonde’s Identity):** For non-negative integers \( m, n, r \) such that \( r \) is the least, \( C(m + n, r) = \sum_{k=0}^{r} C(m, k)C(n, r - k) \).

**Proof:** Consider forming a committee of \( r \geq 0 \) people from a group of \( m \geq r \) men and \( n \geq r \) women. Let \( S \) be the set of all possible committees that can be formed.

The left side, \( C(m + n, r) \), counts how many ways there are to select \( r \) people to from \( m + n \) people. Hence the left side counts \( |S| \).

We can partition \( S \) into \( S_0, S_1, ..., S_r \), where \( S_i \) are the possible committees formed with \( i \) men and \( r - i \) women, for \( i = 0, 1, ..., r \). By the multiplication principle we get that \( |S_i| = C(m, i)C(n, r - i) \) for \( i = 0, 1, ..., r \). Hence the right side, \( \sum_{k=0}^{r} C(m, k)C(n, r - k) \), also counts

\[
|S| = \sum_{i=0}^{r} |S_i| \quad \Box
\]

**Theorem:** For non-negative integers \( n, r \) such that \( r \leq n \), \( C(n + 1, r + 1) = \sum_{k=r}^{n} C(k, r) \).

**Proof:** Consider bit strings (strings of 1’s and 0’s) of length \( n + 1 \) that have exactly \( r + 1 \) 1’s. We will call these \((n+1)\)-bit strings with \( r + 1 \) ones. Let \( S \) be the set of all such strings.

Clearly the left side, \( C(n + 1, r + 1) \), counts \( |S| \).

We can partition \( S \) into \( S_{r+1}, S_{r+2}, ..., S_{n+1} \) where the set \( S_i \) is the subset of \( S \) consisting of \((n+1)\)-bit strings with \( r + 1 \) ones such that the rightmost 1 is in the \( i \)th position (from left to right). For each \( i = r + 1, ..., n + 1 \) we get \( |S_i| = C(i - 1, r) \) (counting combinations of \( r \) ones placed among the first \( i - 1 \) positions, as the \( i \)th position is the rightmost one for the bit strings in \( S_i \)).

Then the right side is equal to \( \sum_{k=r}^{n} C(k, r) = \sum_{i=r+1}^{n+1} C(i - 1, r) = \sum_{i=r+1}^{n+1} |S_i| = |S| \quad \Box \)