Math 254 Week 1 Recitation Activity - Fall 2019

Instructions: Please work in groups for these problems. You should write your solutions on a separate piece of paper when appropriate.

The cross product!
The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is another vector $\vec{u} \times \vec{v}$ with magnitude equal to

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta),$$

where $0 \leq \theta \leq \pi$ is the angle between the two vectors. The direction of $\vec{u} \times \vec{v}$ is given by the right hand rule: when you put the vectors tail to tail and let the fingers of your right hand curl from $\vec{u}$ to $\vec{v}$ the direction of $\vec{u} \times \vec{v}$ is the direction of your thumb, orthogonal to both $\vec{u}$ and $\vec{v}$.

Evaluating the cross product
Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Then $\vec{u} \times \vec{v} = \left\langle +\det \begin{pmatrix} v_2 & v_3 \\ u_2 & u_3 \end{pmatrix}, -\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, +\det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right\rangle$ where $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

(1) Let $\vec{v} = \langle 2, 3, -1 \rangle$ and $\vec{w} = \langle 1, -1, 2 \rangle$. Find the following:

(a) $\vec{v} \cdot \vec{w}$
(b) The angle between $\vec{v}$ and $\vec{w}$
(c) Find $\vec{v} \times \vec{w}$
(d) Find a unit vector orthogonal to both $\vec{v}$ and $\vec{w}$
(e) $\left( \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$
(f) proj$\vec{w} \vec{v}$.

(2) The magnitude of the cross product of vectors $\vec{u}$ and $\vec{v}$ (in 3 dimensions) is the area of parallelogram given by taking two parallel sides of the parallelogram to be $\vec{v}$ and the other two parallel sides to be $\vec{w}$. Find the area of the parallelogram in with vertices at $(1, 2, 3)$, $(3, 3, 4)$, $(5, 2, 2)$ and $(7, 3, 3)$. 

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Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors. For each of the following expressions, identify whether the resulting value is a **scalar**, a **vector**, or the expression is **not defined**. If it is not defined, explain why. If a cross product is involved, assume the vectors are in three dimensions. If a division is involved, assume the denominator is nonzero.

(a) \( \mathbf{u}^2 \)  
(b) \( |\mathbf{u}|^2 \)  
(c) \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)  
(d) \( \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w}) \)  
(e) \( |\mathbf{u}| \times |\mathbf{v}| \)  
(f) \( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \)  
(g) \( \frac{\mathbf{u} \times \mathbf{w}}{\mathbf{v} \cdot \mathbf{w}} \)  
(h) \( \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{v} \times \mathbf{w}} \)

Let \( \vec{v} = 2\hat{i} + 3\hat{j} - \hat{k} \) and \( \vec{w} = \hat{i} - \hat{j} + 2\hat{k} \). Find \( (\vec{v} \times \vec{w}) \cdot \vec{v} \). After you compute this, can you explain the result? What is an easy way to do this problem?

**Definition of Torque:** When a force \( \vec{F} \) is applied to a wrench (or any other lever arm) to tighten or loosen a bolt there is a measurement of twisting energy called **torque**. The torque can be computed with a cross product of \( \vec{r} \) which represents the wrench and \( \vec{F} \) the force applied to the wrench. Then the torque is

\[
\vec{\tau} = \vec{r} \times \vec{F}
\]

and its magnitude (aka **scalar torque**) is

\[
|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin(\theta)
\]

(5) Suppose you apply a force of 20N to a 0.25-meter-long wrench attached to a bolt, at an angle of 45° to the wrench. Determine the magnitude of the torque on the bolt.

(6) Find the equation of the line through \( P(1, 0, 1) \) and \( Q(3, -3, 3) \).

(7) Find an equation of the line through \( P(1, 0, 1) \) parallel to the y-axis.