Show all work clearly and justify your answers using clear logic and complete sentences. You have 35 minutes to take this 10 point quiz. Good luck!

1. (2 points) Determine the truth value of the following statement, with domain of discourse $\mathbb{R} \times \mathbb{R}$. If true, prove it. If false, provide a counterexample.

$$\forall x \forall y ((x < y) \to (x^2 < y^2))$$

Proof. The above statement is false, since if $x = -2, y = 1$, we do indeed have $x < y$, yet $x^2 \not< y^2$. □

2. (3 points) Prove (with a proof) or disprove (with a counter example) the following statement:

$$\exists x \neg(\forall y P(x, y)) \equiv \neg(\forall x \exists y P(x, y))$$

Proof. By DeMorgan’s Laws, we can write the above statement as

$$\exists x \exists y \neg P(x, y) \equiv \exists x \forall y \neg P(x, y).$$

But this is false, since if we consider the counter example: $P(x, y) = ”x < y”$ with $x, y \in \mathbb{Z}$, then the left-hand side of the statement is true (for instance, $x = 1, y = 0$). However, the right-hand side can never be true, since for any $x \in \mathbb{Z}$, we can simply take the value $y = x - 1$ so that $x \not< y$. □
3. (2 points) Disprove the statement: For every positive integer \( n \), \( n^2 \leq 2^n \).

*Proof.* If \( n = 3 \), then \( 3^2 \not\leq 2^3 \). So the statement cannot hold for every positive integer \( n \). \( \square \)

4. (3 points) Show that \( \sqrt[3]{2} \) is irrational. [Hint: Prove this by contradiction.]

*Proof.* Assume for contradiction that \( \sqrt[3]{2} \) is in fact rational. Then there exists \( a, b \in \mathbb{Z} \) with \( b \neq 0 \) and \( a, b \) not sharing any common factors such that

\[
\sqrt[3]{2} = \frac{a}{b}.
\]

By cubing both sides and simplifying, we have

\[
2 = \frac{a^3}{b^3},
\]

hence \( a^3 \) is even. This implies \( a \) is even, since otherwise, if \( a \) were odd, then \( a^3 \) would be odd. To see this, let \( a = 2j + 1 \) for some \( j \in \mathbb{Z} \). Then \( a^3 = (2j + 1)^3 = 8j^3 + 12j^2 + 6j + 1 = 2(4j^3 + 6j^2 + 3j) + 1 \), which is odd, contradicting \( a^3 \) being even. Since \( a \) is now even, set \( a = 2k \) for some \( k \in \mathbb{Z} \). Then

\[
2b^3 = a^3, \\
2b^3 = (2k)^3, \\
2b^3 = 8k^3, \\
b^3 = 4k^3,
\]

hence \( b^3 \) is even. By the previous proof for \( a \) being even, we also have that \( b \) is even. This contradicts \( a, b \) having no common factors. Therefore, \( \sqrt[3]{2} \) must be irrational. \( \square \)