14.8 Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green’s Theorem. Let’s quickly recall that theorem:

**Green’s Theorem (flux form):**

Let \( C \) be a positively oriented (parameterized counterclockwise) piecewise smooth closed simple curve in \( \mathbb{R}^2 \) and \( D \) be the region enclosed by \( C \). Suppose \( \mathbf{F} = \langle f, g \rangle \) where \( f(x, y) \) and \( g(x, y) \) have continuous first order partial derivatives on \( D \). Then

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C -g \, dx + f \, dy = \iint_R \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dA.
\]

One way to think of this is that flux of \( \mathbf{F} \) across the boundary \( C \) of the region \( R \) is equal to the net expansion (which could be contraction) of \( \mathbf{F} \) in the region \( R \).

Now we present the Divergence Theorem:

**Divergence Theorem (aka Gauss’ Theorem):** Let \( \mathbf{F} = \langle f, g, h \rangle \) where \( f, g, h \) have continuous partials inside a simply connected region \( E \) of \( \mathbb{R}^3 \) enclosed by an oriented surface \( S \) in \( \mathbb{R}^3 \). Then the flux surface integral with an outward orientation satisfies

\[
\iint_S \mathbf{F} \cdot dS = \iiint_E (\nabla \cdot \mathbf{F}) \, dV.
\]

The intuition is the same. The flux of \( \mathbf{F} \) across the boundary surface \( S \) of the solid region \( E \) is equal to the net expansion of \( \mathbf{F} \) in \( E \).

Proving this theorem in general is time-consuming, so we just do it for “nice” solids:

Let \( E \) be a “nice” solid (we’ll see what we mean by this later).

Let \( S \) be the boundary of \( E \) and \( \mathbf{n} \) be an outward normal unit vector. Let \( \mathbf{F} = \langle f, g, h \rangle \) have components with continuous partials. Then the surface integral is

\[
\iiint_S \mathbf{F} \cdot dS = \iint_S (fi + gj + hk) \cdot \mathbf{n} \, dS = \iint_S f \mathbf{i} \cdot \mathbf{n} \, dS + \iint_S g \mathbf{j} \cdot \mathbf{n} \, dS + \iint_S h \mathbf{k} \cdot \mathbf{n} \, dS.
\]

While the volume integral is

\[
\iiint_E (\nabla \cdot \mathbf{F}) \, dV = \iiint_E \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) \, dV.
\]
If we can prove that
\[ \int \int \int_S f \cdot n \, dS = \int \int \int_E \frac{\partial f}{\partial x} \, dV, \quad \int \int \int_S g \cdot n \, dS = \int \int \int_E \frac{\partial g}{\partial y} \, dV \text{ and} \]
\[ \int \int \int_S h \cdot n \, dS = \int \int \int_E \frac{\partial h}{\partial z} \, dV, \]
then we will be done.

We prove the third one. The other two are analogous arguments by switching the roles of the variables.

Now the for the “nice” assumption. Let’s assume \( E \) is bounded by two surfaces \( z = p(x, y) \) and \( z = q(x, y) \) over \( R \), the projection of \( E \) in the \( xy \)-plane, such that \( p(x, y) \leq q(x, y) \) on \( R \) (see the picture).

The \( S_1 \) be the graph of \( z = q(x, y) \) and \( S_2 \) be the graph of \( z = p(x, y) \) over \( R \) respectively. Clearly \( S_1 \cup S_2 = S \).

On \( S_1 \) an outward (upward) normal is \( n_1 = (-q_x, -q_y, 1) \), while on \( S_2 \) an outward (downward) normal is \( n_2 = (p_x, p_y, -1) \).
So

\[
\iint_S \mathbf{h} \cdot \mathbf{n} \, dS = \iint_{S_1} \mathbf{h} \cdot \mathbf{n} \, dS + \iint_{S_2} \mathbf{h} \cdot \mathbf{n} \, dS
\]

\[
= \iint_R h(x, y, q(x, y)) \mathbf{k} \cdot \mathbf{n}_1 \, dA + \iint_R h(x, y, p(x, y)) \mathbf{k} \cdot \mathbf{n}_2 \, dA
\]

\[
= \iint_R h(x, y, q(x, y))(+1) \, dA + \iint_R h(x, y, p(x, y))(-1) \, dA
\]

\[
= \iint_R h(x, y, q(x, y)) - h(x, y, p(x, y)) \, dA
\]

\[
= \iint_R \left( \int_{p(x,y)}^{q(x,y)} \frac{\partial h}{\partial z} \, dz \right) \, dA
\]

\[
= \iiint_E \frac{\partial h}{\partial z} \, dV.
\]

And that is what we wanted to show.

Let’s see the usefulness of this theorem!

Let \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) be a fixed non-zero vector in \( \mathbb{R}^3 \). Let

\( \mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_3 y - a_2 z, a_3 x - a_1 z, a_1 y - a_2 x \rangle \) (a rotational vector field).

Let \( S \) be six-sided surface of the cube \( E = [0, 1] \times [0, 1] \times [0, 1] \). Find the (outward) flux \( \mathbf{F} \) through \( S \). This would be rather annoying if we had to do it directly, but by the Divergence Theorem it’s really quite easy:

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV = \iiint_E 0 \, dV = 0.
\]

Note: In this example it doesn’t matter what the solid region is since the vector field is source-free.
Let’s calculate the (outward) flux of $\mathbf{F} = \langle y - z, 2y, x - z \rangle$ through the surface $S$ that bounds the solid $E$ in the first octant cut out by the plane $6x + 2y + z = 12$.

By the Divergence Theorem,

$$\iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV$$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} 1 \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{6-3x} 12 - 6x - 2y \, dy \, dx$$

$$= \int_0^2 (12 - 6x)(6 - 3x) - (6 - 3x)^2 \, dx$$

$$= \int_0^2 72 - 72x + 18x^2 - (36 - 36x + 9x^2) \, dx$$

$$= \int_0^2 36 - 36x + 9x^2 \, dx$$

$$= 36x - 18x^2 + 3x^3 \bigg|_0^2 \, dx$$

$$= 72 - 72 + 24 = 24.$$

That’s much easier than doing a total of 4 surface integrals!

Let’s compute the outward flux of $\mathbf{F} = \langle x, 2y, 3z \rangle$ through the surface $S$ bounding the solid cone $E$ determined by $x^2 + y^2 \leq z^2$ and $0 \leq z \leq 4$.

By the Divergence Theorem,

$$\iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV = \iiint_E 6 \, dV = 6\text{Vol}(E) = 6(1/3)\pi(4^2)(4) = 128\pi.$$

Note: The volume of a cone of radius $R$ and height $H$ is $\frac{1}{3}\pi R^2 H$. 

Let’s compute the outward flux of $\mathbf{F} = \langle xz + e^{yz}, y, z^2 - z + \ln(y^2 + 1) \rangle$ through the surface $S$ that bounds the solid $E = \{(x, y, z) : \sqrt{3} \leq z \leq \sqrt{4 - x^2 - y^2}\}$.

By just looking at the vector field one should hopefully know not to try to do this directly by evaluating surface integrals! We apply the Divergence Theorem:

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_E \nabla \cdot \mathbf{F} \, dV$$

$$= \int_E 3z \, dV$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{3} \sec(\phi)}^2 \rho \cos(\phi) \left( \rho^2 \sin(\phi) \right) d\rho d\phi d\theta$$

$$= 3(2\pi) \int_0^{\pi/6} \int_{\sqrt{3} \sec(\phi)}^2 \rho^3 \sin(\phi) \cos(\phi) \, d\rho d\phi$$

$$= 6\pi \int_0^{\pi/6} \left( \frac{1}{4} \rho^4 \right)_{\sqrt{3} \sec(\phi)}^2 \sin(\phi) \cos(\phi) \, d\phi$$

$$= \frac{3\pi}{2} \int_0^{\pi/6} (16 - 9 \sec^4(\phi)) \sin(\phi) \cos(\phi) \, d\phi$$

$$= \frac{3\pi}{2} \left[ 16 \sin(\phi) \cos(\phi) - \frac{9 \sin(\phi)}{\cos^4(\phi)} \right]_0^{\pi/6}$$

$$= \frac{3\pi}{2} \left( 2 - \frac{9}{2(3/4)} + \frac{9}{2} \right)$$

$$= \frac{3\pi}{4}.$$

That’s still much better than evaluating the surface integrals...
Divergence Theorem for Hollow Regions:

Suppose $E$ is the solid bounded by two oriented surfaces $S_1$ and $S_2$ such that $S_1$ lies within $S_2$. Then the Divergence Theorem becomes

$$\iiint_{E} \nabla \cdot F \, dV = \iint_{S_2} F \cdot dS - \iint_{S_1} F \cdot dS = \iint_{S_1 \cup S_2} F \cdot dS,$$

because of an outward normal of $S_1$ points into $E$.

Let’s find the net outward flux of $F = \langle x^3, y^3, 1 \rangle$ through the surfaces bounding the solid $E$ between the spheres of radii 1 and 2 centered at the origin.

Doing this directly would be time consuming, but by the Divergence Theorem, and with the use of spherical coordinates, the flux is

$$\iiint_{E} \nabla \cdot F \, dV = 3 \iint_{E} x^2 + y^2 \, dV$$

$$= 3 \int_{0}^{2\pi} \int_{0}^{\pi} \int_{1}^{2} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= 6\pi \int_{0}^{\pi} \int_{1}^{2} \rho^4(1 - \cos^2(\phi)) \sin(\phi) d\rho d\phi$$

$$= 6\pi \frac{1}{2} (2^5 - 1) \left[ -\cos(\phi) + \frac{1}{3} \cos^3(\phi) \right]_0^\pi$$

$$= 6\pi \frac{31}{5} \left( \frac{4}{3} \right) = \frac{248\pi}{5}.$$
Gauss’ Law for electric fields states that the electric field due to a point charge \( Q \) is \( \mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} \) where \( \epsilon_0 \) is a constant (known as the permittivity of free space – it measures the resistance to the creation of electric fields).

Let’s directly calculate the outward flux of \( \mathbf{E} \) through the sphere \( S \) of radii \( R \). An outward unit normal vector to \( S \) is \( \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} \); so

\[
\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} dS
\]

\[
= \frac{Q}{4\pi\epsilon_0} \iint_S \frac{1}{|\mathbf{r}|^4} dS
\]

\[
= \frac{Q}{4\pi\epsilon_0} \iint_S \frac{1}{R^2} dS
\]

\[
= \frac{Q}{4\pi R^2\epsilon_0} \iint_S 1 dS
\]

\[
= \frac{Q}{4\pi R^2\epsilon_0} \text{SurfaceArea}(S) = \frac{Q}{\epsilon_0}.
\]

If we naively try to apply the Divergence Theorem, recalling that \( \nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{3-p}{|\mathbf{r}|^p} \), we get

\[
\iint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_{x^2+y^2+z^2 \leq R^2} \nabla \cdot \mathbf{E} dV = \iiint_{x^2+y^2+z^2 \leq R^2} 0 dV = 0.
\]

What’s wrong!?

Well, it’s the partials of the components of the vector field. Since the origin is inside the sphere and the electric field components (and their partials) aren’t continuous at \( \mathbf{0} \), we can’t apply the Divergence Theorem.
Sometimes we can avoid a direct calculation an “ugly” surface integral of a vector field with a “nice” divergence by applying the Divergence Theorem and integrating over an “easy” surface.

Consider having to calculate \( \int \int_{S} F \cdot dS \) where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) with \( z \leq 1 \), oriented outwards and \( F = \langle x^2 + e^z, 1 - xy, x^2 + y^2 \rangle \).

The divergence of \( F \) is \( \nabla \cdot F = x \), which is relatively nice.

Let \( \bar{S} \) be the disk \( \{(x, y, 1) | x^2 + y^2 \leq 3\} \) with an upward orientation. Together \( S \cup \bar{S} \) enclose a solid \( E \) that is symmetric with respect to the \( yz \)-plane. Using the Divergence Theorem, and symmetry we get that
\[
\int \int_{S \cup \bar{S}} F \cdot dS = \iiiint_{E} \nabla \cdot F \, dV = \iiint_{E} x \, dV = 0.
\]

Note: One should be able to compute the integral in spherical coordinates if the symmetry isn’t noticed.

Then \( \int \int_{S \cup \bar{S}} F \cdot dS = \int \int_{S} F \cdot dS + \int \int_{\bar{S}} F \cdot dS = 0 \). Hence \( \int \int_{S} F \cdot dS = - \int \int_{\bar{S}} F \cdot dS \).

A parametrization of \( \bar{S} \) is \( r(u, v) = \langle u \cos(v), u \sin(v), 1 \rangle \) where \( 0 \leq u \leq \sqrt{3} \) and \( 0 \leq v \leq 2\pi \) which leads to \( t_u \times t_v = \langle 0, 0, u \rangle \), which has the correct orientation.

So
\[
\int \int_{S} F \cdot dS = - \int \int_{\bar{S}} F \cdot dS
\]
\[
= - \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} F(r(u, v)) \cdot (t_u \times t_v) \, du \, dv
\]
\[
= - \int_{0}^{2\pi} \int_{0}^{\sqrt{3}} \langle -u, -u^2, 0 \rangle \cdot \langle 0, 0, u \rangle \, du
\]
\[
= -2\pi \int_{0}^{\sqrt{3}} u^2 \, du
\]
\[
= - \frac{9\pi}{2}.
\]
A scalar function $\phi$ is harmonic on a region $D$ in $\mathbb{R}^3$ if $\nabla^2 \phi = \nabla \cdot \nabla \phi = 0$ at all points of $D$.

Let's determine if there are values of $q$ which make $\phi = |r|^{-q}$ harmonic.

Obviously $q = 0$ makes this function constant and therefore harmonic, but what about other values? Well, for $q \neq 0$,

$$\nabla \phi = -q|r|^{-q-1} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = -q|r|^{-q-2}|r| = -q \frac{r}{|r|^{q+2}}.$$  

We want to find values of $q \neq 0$ for which $\nabla \cdot \nabla \phi = 0$:

Since $\nabla \cdot \frac{r}{|r|^p} = 0$ if and only if $p = 3$ it follows that $q + 2 = 3$, or $q = 1$.

So we have that $\phi = 1$ and $\phi = \frac{1}{|r|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ are harmonic.

The flux surface integral of the gradient of a harmonic function $\phi$ is always 0 on a surface $S$ bounding a solid region $E$ (on which $\phi$ harmonic).

$$\int_S \nabla \phi \cdot dS = \int_E \int \nabla \cdot \nabla \phi dV = \int_E \int 0 dV = 0,$$

by applying the Divergence Theorem.

Also, with the assumptions above, the flux surface integral of $\phi \nabla \phi$ satisfies

$$\int_S \phi \nabla \phi \cdot dS = \int_E \int \nabla \cdot (\phi \nabla \phi) dV = \int_E \int \nabla \phi \cdot \nabla \phi + \phi (\nabla \cdot \nabla \phi) \underbrace{dV}_{\text{Zero!}} = \int_E \int |\nabla \phi|^2 dV.$$