14.4 Green’s Theorem

We begin with **Green’s Theorem**:

Let $C$ be a positively oriented (parameterized counterclockwise) piecewise smooth closed simple curve in $\mathbb{R}^2$ and $D$ be the region enclosed by $C$. Suppose $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives on $D$. Then

$$\oint_C P \, dx + Q \, dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

**Note:** $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the two-dimensional curl of the vector field $\mathbf{F} = \langle P, Q \rangle$.

We will give a proof of “half” of Green’s Theorem in the case that the region can be described by the inequalities $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$ (aka a type I region). The proof of the other “half” of Green’s Theorem in the case that the region can be described by inequalities $a \leq y \leq b$ and $f(y) \leq x \leq g(y)$ (aka a type II region) is easily created by adjusting this proof.

Suppose $D$ is described by $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$ (aka a type I region).

Then

$$\int \int_D - \frac{\partial P}{\partial y} \, dA = \oint_{\partial D} P \, dx.$$

**Proof:** Let $D$ be the region bounded by $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$. Let $C = \partial D$ be the boundary of $D$, specifically have $C$ begin at $(a, f(a))$ and go along $y = f(x)$ to $(b, f(b))$, then vertically along $x = b$ to $(b, g(b))$, then along $y = g(x)$ to $(a, g(a))$, and finally vertically along $x = a$ to return to $(a, f(a))$.

$$\int \int_D - \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y} \, dy \, dx = \int_a^b P(x, y) \bigg|_{f(x)}^{g(x)} \, dx =$$

$$= \int_a^b P(x, f(x)) \, dx - \int_a^b P(x, g(x)) \, dx = \int_{C_1} P \, dx + \int_{C_3} P \, dx,$$

where $C_1$ is the curve $y = f(x)$ traversed from $x = a$ to $x = b$ and $C_3$ is the curve $y = g(x)$ traversed from $x = b$ to $x = a$.

Let $C_2$ be the vertical line from $(b, f(b))$ to $(b, g(b))$ and $C_4$ be the vertical line from $(a, g(a))$ to $(a, f(a))$. Since $\frac{dx}{dt} = 0$ for any parametrization $\mathbf{r}(t) = (x(t), y(t))$ along either $C_2$ and $C_4$, we get

$$\int_{C_2} P(x, y) \, dx = \int_{C_2} P(x, y) \frac{dx}{dt} \, dt = 0 \text{ and } \int_{C_4} P(x, y) \, dx = \int_{C_4} P(x, y) \frac{dx}{dt} \, dt = 0.$$
Putting this all together,

\[ \int_C P \, dx = \int_{C_1} P \, dx + \int_{C_2} P \, dx + \int_{C_3} P \, dx + \int_{C_4} P \, dx = \int_{C_1} P \, dx + \int_{C_3} P \, dx = \int_D -\frac{\partial P}{\partial y} \, dA \]

From a similar argument (omitted here) we also get a proof that if a region \( D \) can be described by inequalities \( a \leq y \leq b \) and \( f(y) \leq x \leq g(y) \) (aka a type II region) then

\[ \int \int_D \frac{\partial Q}{\partial x} \, dA = \oint_{\partial D} Q \, dy. \]

These two facts together prove Green’s Theorem in the case that \( D \) is a region that can be described both ways (simultaneously of type I and II).
Any region can be carved by vertical and horizontal lines into subregions that can be described both ways. Here is the region on the previous page divided into 8 such subregions:

For each of these such subregion we have established that Green’s Theorem holds. Now let’s see how that is enough to imply Green’s Theorem for an arbitrary region.

Let $D$ be a region that is “chopped” into subregions $D_1, D_2, ..., D_k$ that are simultaneously of type I and II.

Let $E$ be an “internal” horizontal or vertical edge created. $E$ belongs to exactly two of the subregions $D_i$ and $D_j$. In one of the regions $E$ is either the top (or right) boundary. In the other it is the bottom (or left) boundary. Consequently, $E$ is traversed in one direction using a positive orientation for the boundary of $D_i$ and then traversed in the opposite direction using a positive orientation for the boundary of $D_j$.

Applying Green’s Theorem to each subregion and adding the path integrals over the boundaries of the subregions results in a sum of path integrals that can be rewritten so the path integrals along all internal edges $E$ pair off and cancel, leaving only those along the outer boundary of the entire region (in the positive orientation).

Therefore,

\[
\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \sum_{i=1}^k \int \int_{D_i} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \sum_{i=1}^k \oint_{\partial D_i} P \, dx + Q \, dy = \oint_{\partial D} P \, dx + Q \, dy.
\]
To illustrate, here is an internal edge traversed in opposite directions in this region divided into two regions that are simultaneously of type I and II:

The sum of all the path integrals along the boundaries of the two regions is equal to the path integral around the boundary of the overall region.

Example: Let’s evaluate the following path integral (a) with and (b) without Green’s theorem.

\[ \oint_C x^2 y \, dx - xy \, dy \] where \( C \) is the triangle whose vertices are \((0, 1), (1, 1), \) and \((0, 3)\).

(a) We can break \( C \) into three parts \( C_1, C_2, \) and \( C_3 \) where \( C_1 \) is parameterized by \( \mathbf{r}_1(t) = (t, 1) \) for \( 0 \leq t \leq 1 \), \( \mathbf{r}_2(t) = (1 - t, 1 + 2t) \) for \( 0 \leq t \leq 1 \), and \( \mathbf{r}_3(t) = (0, 3 - t) \) for \( 0 \leq t \leq 2 \).

So \[ \oint_C x^2 y \, dx - xy \, dy = \int_{C_1} x^2 y \, dx - xy \, dy + \int_{C_2} x^2 y \, dx - xy \, dy + \int_{C_3} x^2 y \, dx - xy \, dy = \]

\[ = \int_0^1 t^2 \, dt + \int_0^1 [(1 - t)^2(1 + 2t)(-1) \, dt - (1 - t)(1 + 2t)(2) \, dt] + \int_0^2 0 \, dt = \]

\[ = \frac{1}{3} + \int_0^1 [-(t^2 - 2t + 1)(1 + 2t) - 2(1 + t - 2t^2)] \, dt = \frac{1}{3} + \int_0^1 [-2t^3 + 7t^2 - 2t - 3] \, dt = -\frac{11}{6}. \]
(b) Now let’s use Green’s Theorem!

$$\oint_C x^2y \, dx - xy \, dy = \int \int_D \left[ \frac{\partial}{\partial x} (-xy) - \frac{\partial}{\partial y} (x^2y) \right] \, dA = \int \int_D (-y - x^2) \, dA,$$

where $D$ is the region bounded by \(y = 1, \ x = 0\) and \(y = 3 - 2x\).

Then

$$\oint_C x^2y \, dx - xy \, dy = \int \int_D (-y - x^2) \, dA = \int_0^1 \int_1^{3-2x} (-y - x^2) \, dy \, dx =$$

$$\int_0^1 \left( -\frac{y^2}{2} - x^2y \bigg|_{y=3-2x}^{y=1} \right) \, dx = \int_0^1 \left( \frac{1}{2} - \frac{1}{2}(3 - 2x)^2 + x^2 - x^2(3 - 2x) \right) \, dx =$$

$$\int_0^1 \left( \frac{1}{2} - 2x^2 + 6x - \frac{9}{2} - 2x^2 + 2x^3 \right) \, dx = \int_0^1 \left( -4 + 6x - 4x^2 + 2x^3 \right) \, dx = -\frac{11}{6}.$$

By chopping up the region (as in our justification) Green’s Theorem even works on regions that have “holes” in them (boundary around the “hole” must be oriented negatively).

Evaluate \(\oint_C -y^3 \, dx + x^3 \, dy\) where $C$ is the union of the circles of radii 1 and 2 (centered at the origin).

By Green’s Theorem, \(\oint_C -y^3 \, dx + x^3 \, dy = \int \int_D 3x^2 + 3y^2 \, dA\) where $D$ is the annulus region between the circles of radii 1 and 2 (centered at the origin).

Using polar coordinates, \(\int \int_D 3x^2 + 3y^2 \, dA = \int_0^{2\pi} \int_1^2 3r^2 r \, dr \, d\theta = 2\pi \frac{3}{4} (16 - 1) = \frac{45}{2} \pi.\)
We can use Green’s Theorem to calculate area of a region $R$ enclosed by the closed simple curve $C$:

If $\mathbf{F} = \langle 0, x \rangle$ then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C x \, dy = \int \int_R 1 \, dA = \text{the area of } R.$

Similarly, if $\mathbf{F} = \langle -y, 0 \rangle$ then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y \, dx = \text{the area of } R.$

Let’s calculate the area of portion of the circle of radius 1 that lies above the line $y = \frac{\sqrt{2}}{2}$ using a line integral.

Let $C$ be made up of $C_1$ and $C_2$, where $C_1$ is the part of the unit circle from $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ parameterized by $\mathbf{r}_1 = \langle \cos(t), \sin(t) \rangle$ for $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$ and $C_2$ is the line segment from $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ back to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ parameterized by $\mathbf{r}_2 = \langle t, \frac{\sqrt{2}}{2} \rangle$ for $-\frac{\sqrt{2}}{2} \leq t \leq \frac{\sqrt{2}}{2}$.

Then the area is $A = \oint_C x \, dy = \int_{C_1} x \, dy + \int_{C_2} x \, dy = \int_{\pi/4}^{3\pi/4} \cos(t) \cos(t) \, dt + 0 = \int_{\pi/4}^{3\pi/4} \cos^2(t) \, dt = \frac{1}{2} \int_{\pi/4}^{3\pi/4} 1 + \cos(2t) \, dt = \frac{\pi}{4} - \frac{1}{2}$.

Note: The clever way to solve this is to realize it’s the area of a quarter disk minus a triangle...
Let $C$ be a closed simple curve in $\mathbb{R}^2$ enclosing a region $R$. Let $\mathbf{F} = (f, g)$ be a vector field.

By Green’s Theorem, the flux integral satisfies

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C -g \, dx + f \, dy = \int \int_R \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dA.$$ 

Note: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is the two-dimensional divergence of the vector field $\mathbf{F} = (f, g)$. If the divergence of $\mathbf{F} = (f, g)$ is 0 throughout a region $R$ the vector field $\mathbf{F}$ is said to be source-free.

Example: Use Green’s Theorem to compute the flux integral (outward) of the radial vector field $\mathbf{F} = (x, y)$ across the unit circle $x^2 + y^2 = 1$. As an exercise left for the reader (ELFY), compute it directly as a line integral.

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C -y \, dx + x \, dy = \int \int_{x^2+y^2 \leq 1} (1 + 1) \, dA = 2\pi.$$

Another example: Consider the vector field $\mathbf{F} = (2x + y, x - 4y)$. Let’s calculate the circulation and flux (outward) where $C$ bounds region in the first quadrant between the circles of radii 1 and 2.

Circulation: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (2x + y) \, dx + (x - 4y) \, dy = \int \int_R (1 - 1) \, dA = 0.$

(which is also implied by the fact that this vector field is conservative.)

Flux: $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C -(x - 4y) \, dx + (2x + y) \, dy = \int \int_R (2 - 4) \, dA = -\frac{3\pi}{2}.$

Consider a vector field $\mathbf{F} = (f, g)$, differentiable on a region $R$. A stream function for $\mathbf{F}$ is a function $\psi$ satisfying

$$\psi_y = f \text{ and } \psi_x = -g.$$

If a vector field $\mathbf{F} = (f, g)$ has a stream function $\psi$ (on some region $R$) then the divergence of the vector field is

$$f_x + g_y = (\psi_y)_x + (-\psi_x)_y = \psi_y x - \psi_x y = 0,$$

by the equality of mixed partials.

So the existence of a stream function implies that the field is source-free. The converse turns out to be true (proof skipped) on simply connected regions in $\mathbb{R}^2$. 

Let $\psi$ be a stream function of a source-free vector field $\mathbf{F} = \langle f, g \rangle$. Let’s consider a level curve $C$ of $\psi$ determined by $\psi(x, y) = k$.

Then the gradient $\nabla \psi(x, y) = \langle \psi_x, \psi_y \rangle$ evaluated at a point on $C$ is orthogonal to the level curve $C$. By checking that $\nabla \psi \cdot \mathbf{F} = -gf + fg = 0$ we get that $\mathbf{F}$ must be parallel to $C$ (at points on $C$) since it is also orthogonal to the gradient.

Conclusion: Level curves of stream functions of source-free vector fields are flow curves (aka streamlines) of the vector field.

Consider $\mathbf{F} = \langle x^2, -2xy \rangle$. Since $(x^2)_x + (-2xy)_y = 2x - 2x = 0$ this is a source-free vector field. A stream function for this source-free vector field is $\psi = x^2y$. Let’s compute the flux of this vector field across (a) the unit circle $C$ and (b) across the arc $C_{\alpha\beta}$, the part of the unit circle parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ from $t = \alpha$ to $t = \beta$:

(a) \[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C (-2xy) \, dx + x^2 \, dy = \int_C 2xy \, dx + x^2 \, dy = \int_{x^2+y^2\leq1} (2x - 2x) \, dA = 0.
\]

(b) \[
\int_{C_{\alpha\beta}} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_{\alpha\beta}} (-2xy) \, dx + x^2 \, dy = \int_{C_{\alpha\beta}} 2xy \, dx + x^2 \, dy = \int_{C_{\alpha\beta}} 2 \cos(t) \sin(t) (-\sin(t) \, dt) + \cos^2(t) (\cos(t) \, dt) \\
= \frac{2}{3} \sin^3(t) + \sin(t) - \frac{1}{3} \sin^3(t)
\bigg|_{\alpha}^{\beta} = \sin(t) - \sin(t) \bigg|_{\alpha}^{\beta} = \sin(t) \cos^2(t) \bigg|_{\alpha}^{\beta} = \sin(\beta) \cos^2(\beta) - \sin(\alpha) \cos^2(\beta) = \psi(\mathbf{r}(\beta)) - \psi(\mathbf{r}(\alpha)).
\]

Here we see an important feature of a source-free vector field $\mathbf{F}$ with stream function $\psi$: The flux integral is equal to the value of $\psi$ at the end of the path minus the value of $\psi$ at the start of the path (so is 0 on closed curves). So the flux integral is independent of path for a source-free vector field (similar to the circulation integral being independent of path for a conservative vector field).
Finally, it’s very special when we have a vector field that is BOTH conservative and source-free. Such a vector field is sometimes called an “ideal flow.”

Consider \( \mathbf{F} = \frac{1}{x^2 + y^2} \langle x, y \rangle \). Since

\[
\left( \frac{x}{x^2 + y^2} \right)_y = \frac{-2xy}{(x^2 + y^2)^2} = \left( \frac{y}{x^2 + y^2} \right)_x,
\]

it follows that this vector field is conservative. Since

\[
\left( \frac{x}{x^2 + y^2} \right)_x + \left( \frac{y}{x^2 + y^2} \right)_y = \frac{x^2 + y^2 - 2x^2 + x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = 0,
\]

it follows that this vector field is source-free.

So it follows that the circulation and flux integral of \( \mathbf{F} \) on any closed curve around a region where the components of \( \mathbf{F} \) have continuous partial derivatives (so avoiding the origin) is 0.