14.2 Path and Line Integrals

We begin with a discussion of path integrals (the book calls them scalar line integrals). We will do this for a function of two variables, but these ideas can be naturally applied to a function of \( n \) variables.

Assume \( f(x, y) \) is defined on a smooth curve \( C \) of finite length, parameterized in terms of arc-length as \( r(s) = (x(s), y(s)) \) for \( a \leq t \leq b \). For the purposes of introducing this concept, we assume \( f(x, y) \geq 0 \). We will eventually mention what the difference is if this assumption is taken away.

We partition \( C \) into \( n \) small arcs by forming a partition of \([a, b]\) by picking \( s_i \) as follows:
\[
a = s_0 < s_1 < \cdots < s_n = b.
\]

Let \( \Delta s_i \) for \( i = 1, 2, ..., n \) denote the width of \([s_{i-1}, s_i]\). Pick “sample points \( s^*_i \) from \([s_{i-1}, s_i]\) for \( i = 1, 2, ..., n \).

Then we will approximate the area of the “curtain” under the curve \((x(s), y(s), f(x(s), y(s)))\) (over the curve \( C \) the \( xy \)-plane – see the picture at the top of the next page) by the Riemann sum
\[
A \approx \sum_{k=1}^{n} f(x(s^*_k), y(s^*_k)) \Delta s_k.
\]

Geometrically each summand is the area of a “curvy panel” (to visualize them, take a rectangular piece of paper and bend it).

Let \( \Delta(n) \) be the maximum value among the \( \Delta s_i \) for \( i = 1, 2, ..., n \). If the limit of the Riemann sums as \( n \to \infty \) exists for any sequence of partitions chosen such that \( \Delta(n) \to 0 \) as \( n \to \infty \) then this limit is called the path integral of \( f \) over \( C \) (aka scalar line integral), and it converges to the area of the “curtain.”

See the picture at the top of the next page.

Notation: When the limit exists, we say \( f \) is integrable on the smooth curve \( C \) of finite length and write \( \int_C f \, ds = \int_a^b f(r(s)) \, ds \) (where \( r(s) \) is an arc-length parametrization of \( C \)).

IMPORTANT:

- If we remove the assumption that \( f \) be non-negative, the path integral gives the “net area” (area above the \( xy \)-plane minus area below the \( xy \)-plane).
- This construction and these definitions naturally extend to a scalar-valued function \( f \) on a subset \( S \) of \( \mathbb{R}^n \) containing a smooth curve \( C \) of finite length.
We will compute the area of the “curtain” above, but for now we start with a more simplistic example:

A thin wire in a plane represented by a smooth curve $C$ with a density function $\rho(x,y)$ (units of mass per length) has Mass $M = \int_C \rho \, ds$.

Suppose a thin wire occupies the curve $y = \sqrt{1-x^2}$ and has density $\rho(x,y) = y \text{ g/cm}$ (assume $x,y$ carry units of cm). Let’s find the mass of this wire!

An arc-length parametrization of the upper half of the unit circle is $r(s) = \langle \cos (s), \sin (s) \rangle$ where $0 \leq s \leq \pi$.

So the path integral is

$$M = \int_C \rho \, ds = \int_0^\pi \rho(\cos (s), \sin (s)) \, ds = \int_0^\pi \rho(\cos (s), \sin (s)) \, ds = \int_0^\pi \sin (s) \, ds = 2 \text{ g}.$$
What do we do if we have an arbitrary parametrization of a smooth curve \( C \) of finite length?

We could try to find an arc-length parametrization, but often that creates an extremely “ugly” parametrization. Instead we will develop a formula that is independent of the parametrization!

Let \( C \) a smooth curve of finite length and the vector-valued function \( \mathbf{r}(t) \) for \( a \leq t \leq b \) be an arbitrary differentiable parametrization (moving along the curve in one direction – that is, not back and forth).

Recall the arc-length function below computes the arc-length along \( C \) from \( \mathbf{r}(a) \) to \( \mathbf{r}(t) \):

\[
s(t) = \int_a^t |\mathbf{r}'(u)| \, du.
\]

Differentiating (and using the Fundamental Theorem of Calculus) gives \( s'(t) = |\mathbf{r}'(t)| \). Then we have that \( ds = s'(t) \, dt = |\mathbf{r}'(t)| \, dt \).

Then if \( f \) is integrable, \( \int_C f \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt \). So we can compute the path integral with any parametrization! Naturally we call \( |\mathbf{r}'(t)| \) the speed factor.

Let’s find the area of the curtain at the top of the last page.

Let \( f(x, y) = \frac{3}{2} \cos^{-1}(x) \). Let \( C \) be given by \( \mathbf{r}(t) = (\cos(t/3), \sin(t/3)) \) for \( 0 \leq t \leq 2\pi \). So we find \( \int_C f \, ds \):

\[
\int_C f \, ds = \int_0^{2\pi} \frac{3}{2} \cos^{-1} \left( \cos \left( \frac{t}{3} \right) \right) \left| \frac{1}{3} \left( -\sin \left( \frac{t}{3} \right), \cos \left( \frac{t}{3} \right) \right) \right| \, dt = \frac{1}{2} \int_0^{2\pi} \frac{t}{3} \, dt = \frac{1}{12} (2\pi)^2 = \frac{\pi^2}{3}.
\]

Note: This matches the simple solution which is to realize that this “curtain” is a just a “bent” triangle of base length \( \frac{2\pi}{3} \) and height \( \pi \).
Everything we have done here generalizes for a function of $n$ variables:

Let $f$ be an integrable scalar-valued function on some subset $S$ of $\mathbb{R}^n$ containing a smooth curve $C$ of finite length parameterized by the differentiable vector-valued function

$$r(t) = (x_1(t), x_2(t), ..., x_n(t)) \quad \text{(moving along the curve in one direction)} \quad \text{for} \quad a \leq t \leq b.$$

Then

$$\int_C f \, ds = \int_a^b f(r(t))|r'(t)| \, dt = \int_a^b f(x_1(t), x_2(t), ..., x_n(t)) \sqrt{(x'_1(t))^2 + (x'_2(t))^2 + \cdots + (x'_n(t))^2} \, dt.$$ 

Let’s calculate the path integral $\int_C z^2 \, ds$ for the helical path given by $r(t) = (4 \cos(t), 4 \sin(t), 3t)$ for $0 \leq t \leq \pi/2$.

$$\int_C z^2 \, ds = \int_0^{\pi/2} (3t)^2 \sqrt{(-4 \sin(t))^2 + (4 \cos(t))^2 + (3)^2} \, dt = 45 \int_0^{\pi/2} t^2 \, dt = 15 \left(\frac{\pi}{2}\right)^3 = \frac{15\pi^3}{8}.$$ 

Let $f$ be a function on a subset of $\mathbb{R}^n$ containing a smooth curve $C$ of finite length. The average value of $f$ on $C$ is given by $\frac{1}{\ell(C)} \int_C f \, ds$ where $\ell(C)$ is the length of $C$.

Find the average value of $f(w, x, y, z) = w^2 + x + 2y + 3z$ on the line segment joining $(0, 1, 1, 2)$ to $(1, 2, 2, 1)$.

Let’s find a parametrization of the line segment: $r(t) = (0 + t, 1 + t, 1 + t, 2 - t)$ for $0 \leq t \leq 1$.

The length of this curve is $\ell = \sqrt{1 + 1 + 1 + 1} = 2$. This is also the speed factor!

$$\int_C f \, ds = \int_0^1 (t^2 + (1 + t) + 2(1 + t) + 3(2 - t)) \, (2) \, dt = 2 \int_0^1 t^2 + 9 \, dt = 2 \left(\frac{1}{3} + 9\right) = \frac{56}{3}.$$ 

So the average value of $f$ on the line segment is $\frac{28}{3}$. 
Now we introduce the concept of a line integral of a vector field. First we start with some background.

An oriented curve $C$ is a parameterized curve for which a direction is specified. The positive orientation is the direction the curve is generated as the parameter increases; the negative orientation is direction as the parameter decreases.

For instance, the positive orientation of $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ is counter-clockwise.

Let $\mathbf{F}$ be a continuous vector field on some subset of $\mathbb{R}^n$ that contain a smooth curve $C$ of finite length with an arc-length parametrization $\mathbf{r}(s)$. Let $\mathbf{T}$ be the unit tangent vector at each point of $C$ (with the positive orientation given by the parametrization).

The line integral of $\mathbf{F}$ over a curve $C$ (with a given orientation) is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$.

IMPORTANT: $\mathbf{F} \cdot \mathbf{T}$ is the component of $\mathbf{F}$ in the direction of $\mathbf{T}$. Hence, geometrically, the integral is “adding up” the components of the vector field that are tangential to the curve. As we will see this has many applications.

We immediately seek a way to evaluate a line integral that does not require an arc-length parametrization:

Let $\mathbf{F}$ be a vector field on some subset of $\mathbb{R}^n$ that contains a smooth curve parameterized by a differentiable vector-valued function $\mathbf{r}(t)$ for $a \leq t \leq b$ such that $\mathbf{r}'(t) \neq 0$ for all $t \in [a, b]$.

Then $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$. Then making use of the fact that $ds = |\mathbf{r}'|dt$ we get

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F} \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} |\mathbf{r}'| \, dt = \int_a^b \mathbf{F} \cdot \mathbf{r}' \, dt.$$ 

There are other ways to write this:

$$\int_C \mathbf{F} \cdot d\mathbf{r} \text{ where } d\mathbf{r} = \mathbf{r}' \, dt.$$ 

$$\int_C f_1 \, dx_1 + f_2 \, dx_2 + \cdots + f_n \, dx_n \text{ where } \mathbf{F} = \langle f_1, f_2, \ldots, f_n \rangle \text{ and } \mathbf{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle.$$

Here is the connection: $d\mathbf{r} = \langle x'_1(t), x'_2(t), \ldots, x'_n(t) \rangle \, dt = \langle dx_1, dx_2, \ldots, dx_n \rangle$. 

Let’s work through some examples:

Let’s evaluate the line integral of $\mathbf{F} = \langle x - y, x \rangle$ on the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$:

We can use the graph parametrization $\mathbf{r}(t) = \langle t, t^2 \rangle$ (which has the right orientation). Calling the parabola $C$ we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t - t^2, t \rangle \cdot (1, 2t) dt = \int_0^1 t + t^2 dt = \frac{5}{6}.$$ 

This yields a measure of the strength of the field along the curve in the direction of the curve. Here is a picture:
Work:

If \( \mathbf{F} \) is a vector field in a region \( D \) of \( \mathbb{R}^3 \) representing the force (as a vector) at points inside \( D \) then the work done in moving an object along a curve \( C \) in some direction is the line integral of \( \mathbf{F} \) over \( C \) (using a suitably oriented parametrization).

An electric charge at \((0, 0, 0)\) produces a force field on other point charges given by
\[
\mathbf{F} = \frac{k}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle
\]
where \( k \) is a constant. Let’s calculate the work done in moving a point charge from \((1, 1, 1)\) to \((a, a, a)\):

Using \( \mathbf{r}(t) = \langle t, t, t \rangle \) for \( 1 \leq t \leq a \), we get
\[
W = \int_1^a \frac{k}{(3t^2)^{3/2}} \langle t, t, t \rangle \cdot \langle 1, 1, 1 \rangle \ dt = \int_1^a \frac{k(3t)}{3\sqrt{3}t^3} \ dt = \frac{k}{\sqrt{3}} \int_1^a \frac{1}{t^2} \ dt = \frac{k}{\sqrt{3}} \left( 1 - \frac{1}{a} \right)
\]

Circulation:

Let \( C \) be a closed (ends where it begins) smooth oriented curve in \( \mathbb{R}^n \) parameterized once around by \( \mathbf{r}(t) \) for \( a \leq t \leq b \) (consistent with the orientation) in a region \( D \) where a continuous vector field \( \mathbf{F} \) is defined.

The circulation of \( \mathbf{F} \) on \( C \) is the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

Let’s calculate the circulation of \( \mathbf{F} = \frac{1}{\sqrt{x^2 + y^2}} \langle x, y \rangle \) on the circle \( (x - 2)^2 + (y - 2)^2 = 4 \) using a counter-clockwise orientation:

Let’s label the circle by \( C \). First we parameterize the \( C \) once counterclockwise with \( \mathbf{r}(t) = \langle 2 + 2 \cos(t), 2 + 2 \sin(t) \rangle \) for \( 0 \leq t \leq 2\pi \).

Then we integrate:
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1}{\sqrt{(2 + 2 \cos(t))^2 + (2 + 2 \sin(t))^2}} (2 + 2 \cos(t), 2 + 2 \sin(t)) \cdot (-2 \sin(t), 2 \cos(t)) \ dt
\]
\[
= 2 \int_0^{2\pi} \frac{\cos(t) - \sin(t)}{\sqrt{2 + 2 \cos(t) + 2 \sin(t)}} \ dt
\]
\[
= \sqrt{2} \int_0^{2\pi} \frac{\cos(t) - \sin(t)}{\sqrt{1 + \cos(t) + \sin(t)}} \ dt
\]
\[
= 2\sqrt{2} \sqrt{1 + \cos(t) + \sin(t)} \bigg|_0^{2\pi}
\]
\[
= 0.
\]

The circulation is zero because as much of the vector field points along the path as against the path when traversing \( C \). See the picture on the next page!
Think of it like the vector field is the wind and you walk around $C$. In this case, as much is at your back as at your face...

Let $\mathbf{F} = \langle x^2, 2xy + x, z \rangle$ and let $C$ be the circle $x^2 + y^2 = 1$ oriented counterclockwise in the plane $z = 1$. Let’s find the circulation of $\mathbf{F}$ on $C$:

Let’s use the parametrization $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos^2(t), 2\cos(t)\sin(t) + \cos(t), 1 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin(t)\cos^2(t) + 2\cos^2(t)\sin(t) + \cos^2(t) \, dt$$

$$= \int_0^{2\pi} \cos^2(t)\sin(t) + \cos^2(t) \, dt$$

$$= \int_0^{2\pi} \cos^2(t)\sin(t) + \frac{1}{2} (1 + \cos(2t)) \, dt$$

$$= -\frac{1}{3} \cos^3(t) + \frac{1}{2} \left( t + \frac{1}{2} \sin(2t) \right) \bigg|_0^{2\pi}$$

$$= \pi.$$

Note: Circulation may be computed on a closed curve made up of piecewise smooth curves by just computing the line integral on each curve and adding up the results.
Flux:

The line integral of a vector field $F$ “adds up” the components of $F$ along a smooth curve $C$ of finite length. What if we were to do the same thing, but with the normal components? That is called the flux of $F$ across $C$. We will only focus on how this is developed for a vector field in a plane:

To come up with the flux integral in a plane, we first have to determine a way to find the unit normal vector to a smooth curve $C$ of finite length. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$ be a differentiable parametrization of $C$ (with some fixed orientation) such that $\mathbf{r}' \neq 0$ for all $a \leq t \leq b$.

Recall: $T = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector to $C$ in the direction of the parametrization.

Second we embed the $xy$-plane into $\mathbb{R}^3$, where the vector $\mathbf{k} = \langle 0, 0, 1 \rangle$ is orthogonal to the $xy$-plane. Then a unit vector orthogonal to $C$ in the $xy$-plane would be orthogonal to both $T$ and $\mathbf{k}$. So we set $\mathbf{N} = T \times \mathbf{k}$ (the cross product of two orthogonal unit vectors gives a unit vector orthogonal to both inputs) and drop its third component, which is 0, so that it ultimately lives in $\mathbb{R}^2$.

So the integral for the flux of $F$ across $C$ is $\int_C F \cdot \mathbf{N} \, ds$ where $s$ is arc-length.

IMPORTANT: $F \cdot \mathbf{N}$ is the normal component of $F$ to the curve $C$. In the case where $C$ is a smooth closed curve using a counterclockwise orientation, $\mathbf{N}$ points outward to $C$, and thus, wherever $F$ points outward across $C$ there is a positive contribution to the flux integral, while wherever $F$ point inward across $C$ there is a negative contribution to the flux integral.

Our first goal is to find a nicer formula for flux:

Embedded in $\mathbb{R}^3$, $T = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle x'(t), y'(t), 0 \rangle$.

Then $\mathbf{N} = T \times \mathbf{k} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle y'(t), -x'(t), 0 \rangle$.

After dropping the third component, $\mathbf{N} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle y'(t), -x'(t) \rangle$.

Then with $F = \langle f, g \rangle$

$$\int_C F \cdot \mathbf{N} \, ds = \int_C F \cdot \frac{\langle y'(t), -x'(t) \rangle}{\sqrt{(x'(t))^2 + (y'(t))^2}} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_C f \, dy - g \, dx.$$
Let’s find and simplify the flux integral of \( \mathbf{F} = \frac{1}{\sqrt{x^2 + y^2}} \langle x, y \rangle \) on the circle 

\[(x - 2)^2 + (y - 2)^2 = 4\] using a counter-clockwise orientation:

Let’s label the circle by \( C \). First we parameterize the \( C \) once counterclockwise with \( \mathbf{r}(t) = \langle 2 + 2 \cos(t), 2 + 2 \sin(t) \rangle \) for \( 0 \leq t \leq 2\pi \).

Then we integrate:

\[
\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_0^{2\pi} \frac{(2 + 2 \cos(t))(2 \cos(t)) - (2 + 2 \sin(t))(-2 \sin(t))}{\sqrt{(2 + 2 \cos(t))^2 + (2 + 2 \sin(t))^2}}, \, dt
\]

\[
= 2 \int_0^{2\pi} \frac{1 + \cos(t) + \sin(t)}{\sqrt{2 + 2 \cos(t) + 2 \sin(t)}}, \, dt
\]

\[
= \sqrt{2} \int_0^{2\pi} \frac{1 + \cos(t) + \sin(t)}{\sqrt{1 + \cos(t) + \sin(t)}}, \, dt
\]

\[
= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos(t) + \sin(t)}, \, dt
\]

\[
\approx 7.5.
\]

Note: The integral was approximated numerically with Mathematica.

Anyway, the positive result means that more of the vector field flows out of the region than into the region (just take a look at the picture a couple pages back).