14.7 Stokes’ Theorem

Stokes’ Theorem is the three-dimensional version of the circulation form of Green’s Theorem. Let’s quickly recall that theorem:

**Green’s Theorem:**

Let \( C \) be a positively oriented (parameterized counterclockwise) piecewise smooth closed simple curve in \( \mathbb{R}^2 \) and \( D \) be the region enclosed by \( C \). Suppose \( P(x, y) \) and \( Q(x, y) \) have continuous first order partial derivatives on \( D \). Then

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.
\]

Now we present Stokes’ Theorem:

**Stokes’ Theorem:** Let \( S \) be an oriented surface in \( \mathbb{R}^3 \) with a piecewise-smooth closed boundary \( C \) whose orientation (by the right-hand rule) is consistent with \( S \). Let \( F = \langle f, g, h \rangle \) and assume \( f, g, h \) have continuous partial derivatives on \( S \). Then

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.
\]

Proving Stokes’ Theorem in general is very time-consuming. We just consider a special case:

We consider the case when \( S \) is (part of) the graph of \( z = q(x, y) \) (using the upward orientation). Let \( C \) be the curve that bounds \( S \) with a usual counterclockwise orientation. Let \( R \) be the projection of \( S \) into the \( xy \)-plane and \( C' \) be the projection of \( C \) into the \( xy \)-plane.

Then \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f \, dx + g \, dy + h \, dz \). Using that \( dz = q_x \, dx + q_y \, dy \) we get

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C'} f \, dx + g \, dy + h(q_x \, dx + q_y \, dy) = \oint_{C'} (f + hq_x) \, dx + (g + hq_y) \, dy.
\]

By applying Green’s Theorem above,

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (g + hq_y)_x - (f + hq_x)_y \, dA.
\]

We calculate the partials (where \( f, g, h \) are functions of \((x, y, z)\) and \( z = q(x, y) \)):

\[
(g + hq_y)_x = g_x + g_z q_x + h_x q_y + h_z q_x q_y + h q_y x
\]
and

\[
(f + hq_x)_y = f_y + f_z q_y + h_y q_x + h z q_x q_y + h q_x y.
\]

When we take the difference it simplifies to

\[
(g + hq_y)_x - (f + hq_x)_y = q_x (g_z - h_y) + q_y (h_x - f_z) + (g_x - f_y).
\]
So \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R q_z(g_z - h_y) + q_y(h_x - f_z) + (g_x - f_y) \, dA. \]

Now let’s deal with the surface integral side:

\[ \nabla \times \mathbf{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle \] and we use the graph parametrization. So

\[ \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle \cdot \langle -q_x, -q_y, 1 \rangle \, dA, \]

which is the same as the double integral at the top of this page!

The intuition behind Stokes’ Theorem is the same as for the circulation form of Green’s Theorem: The accumulated rotation of a vector field over a surface \( S \) is the net circulation on the boundary of \( S \), meaning all the internal circulation cancels!

Let’s verify Stokes’ Theorem for \( \mathbf{F} = \langle z - y, x - z, y - x \rangle \) where \( S \) is the cap of \( x^2 + y^2 + z^2 = 4 \) above the plane \( z = \sqrt{3} \) (oriented upwards).

The boundary of \( S \) is the circle \( C = \{(x, y, z)|x^2 + y^2 = 1, z = \sqrt{3}\} \). A parametrization of that circle (consistent with the upward orientation of \( S \) is \( \mathbf{r}(t) = \langle \cos(t), \sin(t), \sqrt{3} \rangle \). So

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
= \int_0^{2\pi} \langle \sqrt{3} - \sin(t), \cos(t) - \sqrt{3}, \sin(t) - \cos(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt \\
= \int_0^{2\pi} -\sqrt{3} \sin(t) + \sin^2(t) + \cos^2(t) - \sqrt{3} \cos(t) \, dt \\
= 2\pi. \]

\[ \nabla \times \mathbf{F} = \langle z - y, x - z, y - x \rangle = \langle 2, 2, 2 \rangle. \]

Let’s use the parametrization \( \mathbf{r}(u, v) = 2\langle \sin(u) \cos(v), \sin(u) \cos(v), \cos(u) \rangle \) over the rectangle \( R \) given by \( 0 \leq u \leq \frac{\pi}{6} \) and \( 0 \leq v \leq 2\pi \). This gives

\[ \mathbf{t}_u \times \mathbf{t}_v = 4\langle \sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u) \rangle \]
So

\[ \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_R \langle 2, 2, 2 \rangle \cdot 4\langle \sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u) \rangle \, dA \]
\[ = 8 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^2(u) \cos(v) + \sin^2(u) \sin(v) + \sin(u) \cos(u) \, dv \, du \]
\[ = 8 \int_0^{\frac{\pi}{2}} 2\pi \sin(u) \cos(u) \, du \]
\[ = 8\pi \sin^2(u) \bigg|_0^{\frac{\pi}{2}} \]
\[ = 2\pi. \]

And that verifies Stokes’ Theorem in this instance!

Now let’s see the usefulness of Stokes’ Theorem:

Let \( \mathbf{F} = \langle y^2, -x^2, x+y+z \rangle \). Let’s evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the boundary of the square \( S \)

in the \( xy \)-plane given by \(|x| \leq 1 \) and \(|y| \leq 1 \) oriented consistent with an upward orientation on the square.

Doing this directly would require parameterizing the four line segments (in a counterclockwise orientation) that make up the square and evaluating the line integral on each. By Stokes’ Theorem we can make this much easier:

\[ \nabla \times \mathbf{F} = \langle 1, -1, -2x - 2y \rangle. \]

Using the parametrization \( \mathbf{s}(x,y) = \langle x, y, 0 \rangle \) over the square \([-1, 1] \times [-1, 1] \) yields \( d\mathbf{S} = \langle 0, 0, 1 \rangle \, dA \) (obviously?) and hence

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_R \langle 1, -1, -2x - 2y \rangle \cdot \langle 0, 0, 1 \rangle \, dA = \int_{-1}^{1} \int_{-1}^{1} -2x - 2y \, dy \, dx = 0. \]

If you are bored, come up with the same answer by evaluating \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) directly.
Let $F = \langle z, x, y \rangle$. Let’s evaluate $\iint_S (\nabla \times F) \cdot dS$ where $S$ is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$, for $z \geq 0$ (with an upward orientation).

Let $C$ be the boundary of $S$ in the plane $z = 0$: So $C$ is the circle of radius $3\sqrt{11}$ in the plane $z = 0$ centered at the origin. Then $r(t) = 3\sqrt{11}\langle \cos (t), \sin (t), 0 \rangle$ for $0 \leq t \leq 2\pi$ is a parametrization of $C$ (counterclockwise). Hence

$$\iint_S (\nabla \times F) \cdot dS = \oint_C F \cdot dr = \int_0^{2\pi} (0, 3\sqrt{11}\cos (t), 3\sqrt{11}\sin (t)) \cdot 3\sqrt{11}(-\sin (t), \cos (t), 0) \, dt = 99 \int_0^{2\pi} \cos^2 (t) \, dt = 99 \int_0^{2\pi} (1/2)(1 + \cos (2t)) \, dt = 99\pi.$$

Just for fun, let’s try to do this calculation without Stokes’ Theorem:

$\nabla \times F = \langle 1, 1, 1 \rangle$

We could use the graph parametrization, but alternatively consider having $x^2 + y^2 = v^2$.

So let’s use the parametrization $r(u, v) = \langle v \cos (u), v \sin (u), 10 - \sqrt{1 + v^2} \rangle$ over the rectangle $R$ determined by $0 \leq u \leq 2\pi$ and $0 \leq v \leq 3\sqrt{11}$.

Then $t_u = \langle -v \sin (u), v \cos (u), 0 \rangle$

and $t_v = \langle \cos (u), \sin (u), -\frac{v}{\sqrt{1 + v^2}} \rangle$.

So $t_u \times t_v = \langle -\frac{v^2 \cos (u)}{\sqrt{1 + v^2}}, -\frac{v^2 \sin (u)}{\sqrt{1 + v^2}}, -v \rangle$.

Since this points downward, we will use $-t_u \times t_v$:

$$\iint_S (\nabla \times F) \cdot dS \quad = \quad \iint_R \langle 1, 1, 1 \rangle \cdot \left\langle \frac{v^2 \cos (u)}{\sqrt{1 + v^2}}, \frac{v^2 \sin (u)}{\sqrt{1 + v^2}}, v \right\rangle \, dA$$

$$= \int_0^{3\sqrt{11}} \int_0^{2\pi} \frac{v^2}{\sqrt{1 + v^2}}(\cos (u) + \sin (u)) + v \, du \, dv$$

$$= \int_0^{3\sqrt{11}} 0 + 2\pi v \, dv$$

$$= 99\pi.$$
Let $D$ be the “tilted” disk enclosed by the curve $C$ given by $r(t) = \langle \cos (\phi) \cos (t), \sin (t), \sin (\phi) \cos (t) \rangle$ where $0 \leq \phi \leq \pi/2$ is a fixed angle (between the disk and the positive $x$-axis).

Let’s find the area of $D$:

$$\text{Area}(D) = \iint_D 1 \, dS.$$ 

Let’s find a normal vector to $D$: $r(0) \times r'(0) = \langle \cos (\phi), 0, \sin (\phi) \rangle \times \langle 0, 1, 0 \rangle = \langle -\sin (\phi), 0, \cos (\phi) \rangle$. We set this (unit vector) equal to $n$.

By finding a vector field $F$ such that $\nabla \times F = \langle -\sin (\phi), 0, \cos (\phi) \rangle$ we will have $(\nabla \times F) \cdot n = 1$.

Well, $F = \langle -y \cos (\phi), 0, -y \sin (\phi) \rangle$ does the trick! Thus

$$\text{Area}(D) = \iint_D 1 \, dS \quad = \iint_D (\nabla \times F) \cdot n \, dS \quad = \oint_C F \cdot dr \quad = \int_0^{2\pi} \langle -\sin (t) \cos (\phi), 0, -\sin (t) \sin (\phi) \rangle \cdot \langle -\cos (\phi) \sin (t), \cos (t), -\sin (\phi) \sin (t) \rangle \, dt \quad = \int_0^{2\pi} \sin^2 (t) \cos^2 (\phi) + \sin^2 (t) \sin^2 (\phi) \, dt \quad = \int_0^{2\pi} \sin^2 (t) \, dt \quad = \int_0^{2\pi} (1/2)(1 - \cos (2t)) \, dt \quad = \pi.$$

Of course!
Now let’s find the circulation \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) for \( \mathbf{F} = (-y, x, 0) \) as a function of \( \phi \). Then let’s determine the value of \( \phi \) for which the circulation is at a maximum.

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

\[
= \iint_D \langle 0, 0, 2 \rangle \cdot \langle -\sin(\phi), 0, \cos(\phi) \rangle \, dS
\]

\[
= 2 \cos(\phi) \iint_D 1 \, dS
\]

\[
= 2\pi \cos(\phi).
\]

This is at a maximum when \( \phi = 0 \) and \( D \) is the unit disk in the \( xy \)-plane.

Consider the vector field \( \mathbf{F} = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right\rangle \).

A quick calculation, which is left as an exercise for you, shows that \( \nabla \times \mathbf{F} = 0 \).

This would seem to imply that for any closed simple curve \( C \) in the \( xy \)-plane encircling a region \( S \) containing the origin,

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.
\]

However, let’s see what happens when we calculate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) directly for the unit circle \( C \) in the \( xy \)-plane, which is parameterized by \( \mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle \) for \( 0 \leq t \leq 2\pi \):

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t), 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt = 2\pi \neq 0.
\]

What’s wrong? Is this a violation of Stokes’ Theorem?!

Of course it isn’t. Stokes’ Theorem doesn’t apply here because the component functions of \( \mathbf{F} \) aren’t continuous at the origin, let alone have continuous partial derivatives at the origin.