14.7 Stokes’ Theorem

Stokes’ Theorem is the three-dimensional version of the circulation form of Green’s Theorem. Let’s quickly recall that theorem:

Green’s Theorem:
Let $C$ be a positively oriented (parameterized counterclockwise) piecewise smooth closed simple curve in $\mathbb{R}^2$ and $D$ be the region enclosed by $C$. Suppose $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives on $D$. Then

$$\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Now we present Stokes’ Theorem:

Stokes’ Theorem: Let $S$ be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary $C$ whose orientation (by the righthand rule) is consistent with $S$. Let $F = \langle f, g, h \rangle$ and assume $f, g, h$ have continuous partial derivatives on $S$. Then

$$\oint_C F \cdot d\mathbf{r} = \iint_S (\nabla \times F) \cdot d\mathbf{S}.$$

Proving Stokes’ Theorem in general is very time-consuming. We just consider a special case:

We consider the case when $S$ is (part of) the graph of $z = q(x, y)$ (using the upward orientation). Let $C$ be the curve that bounds $S$ with a usual counterclockwise orientation. Let $R$ be the projection of $S$ into the $xy$-plane and $C'$ be the projection of $C$ into the $xy$-plane.

Then $\oint_C F \cdot d\mathbf{r} = \oint_C f \, dx + g \, dy + h \, dz$. Using that $dz = q_x \, dx + q_y \, dy$ we get

$$\oint_C F \cdot d\mathbf{r} = \oint_{C'} f \, dx + g \, dy + h(q_x \, dx + q_y \, dy) = \oint_{C'} (f + h q_x) \, dx + (g + h q_y) \, dy.$$

By applying Green’s Theorem above,

$$\oint_C F \cdot d\mathbf{r} = \iint_R (g + h q_y) \, dx - (f + h q_x) \, dy \, dA.$$

We calculate the partials (where $f, g, h$ are functions of $(x, y, z)$ and $z = q(x, y)$):

$(g + h q_y)_x = g_x + g_z q_x + h_x q_y + h_z q_x q_y + h q_{yx}$ and

$(f + h q_x)_y = f_y + f_z q_y + h_y q_x + h_z q_y q_x + h q_{xy}$.

When we take the difference it simplifies to

$(g + h q_y)_x - (f + h q_x)_y = q_x (g_z - h_y) + q_y (h_x - f_z) + (g_x - f_y)$. 
So $\int_C \mathbf{F} \cdot d\mathbf{r} = \iiint_R q_x (g_z - h_y) + q_y (h_x - f_z) + (g_x - f_y) \, dA$.

Now let’s deal with the surface integral side:

$\nabla \times \mathbf{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$ and we use the graph parametrization. So

$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle \cdot \langle -q_x, -q_y, 1 \rangle \, dA,$

which is the same as the double integral at the top of this page!

The intuition behind Stokes’ Theorem is the same as for the circulation form of Green’s Theorem: The accumulated rotation of a vector field over a surface $S$ is the net circulation on the boundary of $S$, meaning all the internal circulation cancels!

Let’s verify Stokes’ Theorem for $\mathbf{F} = \langle z - y, x - z, y - x \rangle$ where $S$ is the cap of $x^2 + y^2 + z^2 = 4$ above the plane $z = \sqrt{3}$ (oriented upwards).

The boundary of $S$ is the circle $C = \{ (x, y, z) | x^2 + y^2 = 1, z = \sqrt{3} \}$. A parametrization of that circle (consistent with the upward orientation of $S$) is $\mathbf{r}(t) = \langle \cos (t), \sin (t), \sqrt{3} \rangle$. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{2\pi} \langle \sqrt{3} - \sin (t), \cos (t) - \sqrt{3}, \sin (t) - \cos (t) \rangle \cdot \langle -\sin (t), \cos (t), 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sqrt{3} \sin (t) + \sin^2 (t) + \cos^2 (t) - \sqrt{3} \cos (t) \, dt$$

$$= 2\pi.$$

$\nabla \times \mathbf{F} = \langle z - y, x - z, y - x \rangle = \langle 2, 2, 2 \rangle$.

Let’s use the parametrization $\mathbf{r}(u, v) = 2 \langle \sin (u) \cos (v), \sin (u) \cos (v), \cos (u) \rangle$ over the rectangle $R$ given by $0 \leq u \leq \frac{\pi}{6}$ and $0 \leq v \leq 2\pi$. This gives

$\mathbf{t}_u \times \mathbf{t}_v = 4 \langle \sin^2 (u) \cos (v), \sin^2 (u) \sin (v), \sin (u) \cos (u) \rangle$
\[ \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_R \langle 2, 2, 2 \rangle \cdot 4(\sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u)) \, dA \]
\[ = 8 \int_0^{\pi/6} \int_0^{2\pi} \sin^2(u) \cos(v) + \sin^2(u) \sin(v) + \sin(u) \cos(u) \, dv \, du \]
\[ = 8 \int_0^{\pi/6} 2\pi \sin(u) \cos(u) \, du \]
\[ = 8\pi \sin^2(u) \bigg|_0^{\pi/6} \]
\[ = 2\pi. \]

And that verifies Stokes’ Theorem in this instance!

Now let’s see the usefulness of Stokes’ Theorem:

Let \( \mathbf{F} = \langle y^2, -x^2, x+y+z \rangle \). Let’s evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the boundary of the square \( S \) in the \( xy \)-plane given by \( |x| \leq 1 \) and \( |y| \leq 1 \) oriented consistent with an upward orientation on the square.

Doing this directly would require parameterizing the four line segments (in a counterclockwise orientation) that make up the square and evaluating the line integral on each. By Stokes’ Theorem we can make this much easier:

\[ \nabla \times \mathbf{F} = \langle 1, -1, -2x-2y \rangle. \] Using the parametrization \( \mathbf{s}(x, y) = \langle x, y, 0 \rangle \) over the square \([0, 1] \times [0, 1]\) yields \( d\mathbf{S} = \langle 0, 0, 1 \rangle \, dA \) (obviously?) and hence

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_R \langle 1, -1, -2x-2y \rangle \cdot \langle 0, 0, 1 \rangle \, dA = \int_0^1 \int_0^1 -2x-2y \, dy \, dx = -2. \]

If you are bored, come up with the same answer by evaluating \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) directly.
Let $F = \langle z, x, y \rangle$. Let's evaluate $\int \int_S (\nabla \times F) \cdot dS$ where $S$ is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$, for $z \geq 0$ (with an upward orientation).

Let $C$ be the boundary of $S$ in the plane $z = 0$: So $C$ is the circle of radius 3 in the plane $z = 0$ centered at the origin. Then $r(t) = 3(\cos(t), \sin(t), 0)$ for $0 \leq t \leq 2\pi$ is a parametrization of $C$ (counterclockwise). Hence

$$\int \int_S (\nabla \times F) \cdot dS = \oint_C F \cdot dr = \int_0^{2\pi} \langle 0, 3\cos(t), 3\sin(t) \rangle \cdot 3\langle -\sin(t), \cos(t), 0 \rangle dt =$$

$$= 9 \int_0^{2\pi} \cos^2(t) \, dt = 9 \int_0^{2\pi} (1/2)(1 + \cos(2t)) \, dt = 9\pi.$$

Just for fun, let's try to do this calculation without Stokes' Theorem:

$$\nabla \times F = \langle 1, 1, 1 \rangle$$

We could use the graph parametrization, but alternatively consider having $x^2 + y^2 = v^2$.

So let's use the parametrization $r(u, v) = \langle v \cos(u), v \sin(u), 10 - \sqrt{1 + v^2} \rangle$ over the rectangle $R$ determined by $0 \leq u \leq 2\pi$ and $0 \leq v \leq 3$.

Then $t_u = \langle -v \sin(u), v \cos(u), 0 \rangle$ and $t_v = \langle \cos(u), \sin(u), -\frac{v}{\sqrt{1 + v^2}} \rangle$.

So $t_u \times t_v = \langle -\frac{v^2 \cos(u)}{\sqrt{1 + v^2}}, -\frac{v^2 \sin(u)}{\sqrt{1 + v^2}}, -v \rangle$.

Since this points downward, we will use $-t_u \times t_v$:

$$\int \int_S (\nabla \times F) \cdot dS = \int \int_R \langle 1, 1, 1 \rangle \cdot \left( \frac{v^2 \cos(u)}{\sqrt{1 + v^2}}, \frac{v^2 \sin(u)}{\sqrt{1 + v^2}}, v \right) \, dA$$

$$= \int_0^3 \int_0^{2\pi} \frac{v^2}{\sqrt{1 + v^2}} (\cos(u) + \sin(u)) + v \, du \, dv$$

$$= \int_0^3 0 + 2\pi v \, dv$$

$$= 9\pi.$$
Let $D$ be the “tilted” disk enclosed by the curve $C$ given by
$r(t) = \langle \cos(\phi) \cos(t), \sin(t), \sin(\phi) \cos(t) \rangle$ where $0 \leq \phi \leq \pi/2$ is a fixed angle (between the disk and the positive $x$-axis).

Let’s find the area of $D$:

$$\text{Area}(D) = \int\int_D 1 \, dS.$$ 

Let’s find a normal vector to $D$: $r(0) \times r'(0) = \langle \cos(\phi), 0, \sin(\phi) \rangle \times \langle 0, 1, 0 \rangle = \langle -\sin(\phi), 0, \cos(\phi) \rangle$. We set this (unit vector) equal to $n$.

By finding a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = \langle -\sin(\phi), 0, \cos(\phi) \rangle$ we will have $(\nabla \times \mathbf{F}) \cdot n = 1$.

Well, $\mathbf{F} = \langle -y \cos(\phi), 0, -y \sin(\phi) \rangle$ does the trick! Thus

$$\text{Area}(D) = \int\int_D 1 \, dS$$

$$= \int\int_D (\nabla \times \mathbf{F}) \cdot n \, dS$$

$$= \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \langle -\sin(t) \cos(\phi), 0, -\sin(t) \sin(\phi) \rangle \cdot \langle -\cos(\phi) \sin(t), \cos(t), -\sin(\phi) \sin(t) \rangle \, dt$$

$$= \int_0^{2\pi} \sin^2(t) \cos^2(\phi) + \sin^2(t) \sin^2(\phi) \, dt$$

$$= \int_0^{2\pi} \sin^2(t) \, dt$$

$$= \int_0^{2\pi} (1/2)(1 - \cos(2t)) \, dt$$

$$= \pi.$$ 

Of course!
Now let’s find the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle -y, x, 0 \rangle$ as a function of $\phi$. Then let’s determine the value of $\phi$ for which the circulation is at a maximum.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

$$= \iint_D \langle 0, 0, 2 \rangle \cdot \langle -\sin (\phi), 0, \cos (\phi) \rangle \, dS$$

$$= 2 \cos (\phi) \iint_D 1 \, dS$$

$$= 2\pi \cos (\phi).$$

This is at a maximum when $\phi = 0$ and $D$ is the unit disk in the $xy$-plane.

Consider the vector field $\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right\rangle$.

A quick calculation, which is left as an exercise for you, shows that $\nabla \times \mathbf{F} = 0$.

This would seem to imply that for any closed simple curve $C$ in the $xy$-plane encircling a region $S$ containing the origin,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$  

However, let’s see what happens when we calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly for the unit circle $C$ in the $xy$-plane, which is parameterized by $\mathbf{r}(t) = \langle \cos (t), \sin (t), 0 \rangle$ for $0 \leq t \leq 2\pi$:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin (t), \cos (t), 0 \rangle \cdot \langle -\sin (t), \cos (t), 0 \rangle \, dt = 2\pi \neq 0.$$  

What’s wrong? Is this a violation of Stokes’ Theorem!?  

Of course it isn’t. Stokes’ Theorem doesn’t apply here because the component functions of $\mathbf{F}$ aren’t continuous at the origin, let alone have continuous partial derivatives at the origin.