14.6 Surface Integrals

Let’s develop some surface integrals.

First we consider how to parameterize a surface (similar to a parameterized curve for line integrals). Surfaces will need two parameters.

With \( u, v \) as the parameters, a parametrization of surface \( S \) in \( \mathbb{R}^3 \) has the form

\[
\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,
\]

where the range of \( \mathbf{r} \) (on some rectangular domain \( R \) in the \( uv\)-plane) is \( S \).

Examples:

1. \( \mathbf{r}(u, v) = \langle R \cos (u), R \sin (u), v \rangle \) on the rectangle given by \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq H \).

   This is a parametrization of the curved surface of a right circular cylinder of radius \( R \) about the \( z \)-axis from \( z = 0 \) to \( z = H \) (so the height is \( H \)).

2. \( \mathbf{r}(u, v) = \langle Rv \cos (u)/H, Rv \sin (u)/H, v \rangle \) on the rectangle given by \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq H \).

   Here we have circular traces (at a fixed \( z = v \)) that vary in radii from 0 at \( z = 0 \) to \( R \) at \( z = H \). Hence this is a parametrization of the curved surface of a right circular inverted cone of radius \( R \) from \( z = 0 \) (at the vertex) to \( z = H \) (so the height is \( H \)).

3. \( \mathbf{r}(u, v) = \langle \rho \sin (u) \cos (v), \rho \sin (u) \sin (v), \rho \cos (u) \rangle \) on the rectangle given by \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \).

   Spherical coordinates gives us this one! Here \( u \) is playing the role of \( \phi \) and \( v \) is playing the role of \( \theta \). This is a parametrization of the sphere of radius \( \rho \) centered at the origin.

4. Suppose we want to parameterize some part of the graph of \( z = f(x, y) \) above a rectangle \( R \) in the \( xy\)-plane. Then we use the graph parametrization:

   \( \mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle \) over the rectangle \( R \) in the \( uv\)-plane.
Surface integrals of scalar functions:

Suppose we have a scalar function \( f(x, y, z) \) defined on a smooth surface \( S \) in \( \mathbb{R}^3 \) parameterized by \( \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \), over a rectangle \( R \) given by \( a \leq u \leq b \) and \( c \leq v \leq d \), where the functions \( x(u, v) \), \( y(u, v) \) and \( z(u, v) \) are assumed to have continuous partials.

Partition the rectangle \( R \) into \( n \) subrectangles (ordered by \( k = 1, 2, ..., n \)) with sides of length \( \Delta u \) and \( \Delta v \). The area of each rectangle is \( \Delta A = \Delta u \Delta v \). The \( k \)th rectangle, \( R_k \), with area \( \Delta A \) corresponds under \( \mathbf{r} \) to a curved patch \( S_k \) of the surface \( S \) with area \( \Delta S_k \). Let \( (u_k, v_k) \) be the lower left corner of rectangle \( R_k \). Let \( x_k = x(u_k, v_k), y_k = y(u_k, v_k) \) and \( z_k = z(u_k, v_k) \).

From this we derive a Riemann sum, which adds up the function’s values multiplied by the areas of the surface patches:

\[
\sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta S_k.
\]

In order to be able to turn this into an integral (by having \( n \to \infty \)) which we can evaluate, we need to express \( \Delta S_k \) in terms of \( \Delta A \).

Let \( t_u = \frac{\partial \mathbf{r}}{\partial u} \) and \( t_v = \frac{\partial \mathbf{r}}{\partial v} \). When evaluated at \( (u_k, v_k) \) these are tangent vectors to the surface at \( (x_k, y_k, z_k) \) in the patch of the surface (in directions given by varying one of \( u \) or \( v \) while holding the other one fixed).

We want to consider the tangent vectors \( t_u \Delta u \) and \( t_v \Delta v \) so that as \( n \to \infty \) these vectors get arbitrarily small. These vectors are approximations of \( \Delta \mathbf{r} \) (in the respective directions) along two sides of the patch \( S_k \), so the magnitude of their cross product is the area of a parallelogram that is a decent approximation to \( \Delta S_k \):

\[
\Delta S_k \approx |t_u \Delta u \times t_v \Delta v| = |t_u \times t_v| \Delta u \Delta v.
\]

See picture on next page for a visualization.

Note: The vector \( t_u \times t_v \) is orthogonal (aka normal) to the surface at \( (x_k, y_k, z_k) \). The factor \( |t_u \times t_v| \) is similar to the speed factor in a path/line integral.

So the Riemann sum becomes:

\[
\sum_{k=1}^{n} f(x_k, y_k, z_k)|t_u \times t_v| \Delta A.
\]

This gives us the integral of \( f \) over the surface \( S \), denoted \( \iint_S f \, dS \), by taking the limit as \( n \to \infty \):

\[
\iint_S f \, dS = \iint_R f(x(u, v), y(u, v), z(u, v))|t_u \times t_v| \, dA.
\]

Note: If \( f = 1 \) then the integral is the area of the surface \( S \).
Let’s use a surface integral to find the surface area of the sphere $S$ of radius $R$:

Let’s use the parametrization $\mathbf{r}(u, v) = \langle R \sin (u) \cos (v), R \sin (u) \sin (v), R \cos (u) \rangle$, where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

Then $\mathbf{t}_u = R \langle \cos (u) \cos (v), \cos (u) \sin (v), -\sin (u) \rangle$ and $\mathbf{t}_v = R \langle \sin (u) \sin (v), \sin (u) \cos (v), 0 \rangle$. So

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix} = -R^2 \langle \sin^2 (u) \cos (v), \sin^2 (u) \sin (v), \sin (u) \cos (u) \rangle.$$

Then $|\mathbf{t}_u \times \mathbf{t}_v| = R^2 \sqrt{(\sin^2 (u) \cos (v))^2 + (\sin^2 (u) \sin (v))^2 + (\sin (u) \cos (u))^2}$

$= R^2 \sqrt{\sin^4 (u) \cos^2 (v) + \sin^4 (u) \sin^2 (v) + \sin^2 (u) \cos^2 (u)}$

$= R^2 \sqrt{\sin^4 (u) + \sin^2 (u) \cos^2 (u)}$

$= R^2 \sin (u)$ using that $\sin (u) \geq 0$ on $[0, \pi]$.

Note: This is the Jacobian of spherical coordinates where $\rho = R$!

So the sphere’s area is

$$\int_S 1 \, dS = \int_0^{2\pi} \int_0^\pi R^2 \sin (u) \, du \, dv = 2\pi R^2 (-\cos (u))^{\pi} = 4\pi R^2.$$

Let’s evaluate the surface integral of $f(x, y) = x^2 + y^2$ over the top hemisphere $H = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \geq 0 \}$ of radius 1. We can use the parametrization above with $R = 1$ and the restriction $0 \leq u \leq \pi/2$ to get straight to it:

$$\int_H x^2 + y^2 \, dS = \int_0^{2\pi} \int_0^{\pi/2} \left[ (\sin (u) \cos (v))^2 + (\sin (u) \sin (v))^2 \right] (\sin (u)) \, du \, dv$$

$= 2\pi \int_0^{\pi/2} \sin^3 (u) \, du$

$= 2\pi \int_0^{\pi/2} (1 - \cos^2 (u)) \sin (u) \, du$

$= 2\pi \int_1^0 1 - x^2 \, (-dx)$

$= 2\pi \int_0^1 1 - x^2 \, dx$

$= 2\pi (1 - 1/3) = \frac{4\pi}{3}.$

Since the area of $H$ is $2\pi$ it follows that the average value of $x^2 + y^2$ on $H$ is $2/3$. 
Let’s develop a formula that will work for the surface integral of \( f(x, y, z) \) on a surface \( S_g \) determined by the graph of \( z = g(x, y) \):

We begin with the graph parametrization. \( \mathbf{r} = \langle u, v, g(u, v) \rangle \) on a region \( R \) (at this point we drop the assumption that \( R \) must be a rectangle, the construction still holds for more general regions).

Then \( \mathbf{t}_u = \langle 1, 0, g_u \rangle \) and \( \mathbf{t}_v = \langle 0, 1, g_v \rangle \) so it follows that \( \mathbf{t}_u \times \mathbf{t}_v = \langle -g_u, -g_v, 1 \rangle \). Hence, \( |\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{(g_u)^2 + (g_v)^2 + 1} \).

Thus \( \int \int_{S_g} f \ dS = \int \int_{R} f(u, v, g(u, v)) \sqrt{(g_u)^2 + (g_v)^2 + 1} \ dA. \)

We could leave things in terms of \( x \) and \( y \) instead of \( u \) and \( v \):

\[ \int \int_{S_g} f \ dS = \int \int_{R} f(x, y, g(x, y)) \sqrt{(g_x)^2 + (g_y)^2 + 1} \ dA. \]

Let’s find the surface integral of \( f(x, y, z) = e^z \) over the surface \( S \) that is the part of the plane \( 3x + 2y + z = 6 \) in the first octant.

The function \( g(x, y) \) in this case is \( z = g(x, y) = 6 - 3x - 2y \). So this surface integral is

\[ \int \int_{S} f \ dS = \int \int_{R} e^{6 - 3x - 2y} \sqrt{4 + 9 + 1} \ dA, \]

where \( R \) is the region in the first quadrant bounded by \( 3x + 2y = 6 \).

So

\[
\int \int_{S} f \ dS = \int_{0}^{2} \int_{0}^{\frac{6-3x}{2}} e^{6} e^{-3x} e^{-2y} \sqrt{14} \ dy \ dx = \frac{1}{2} e^{6} \sqrt{14} \int_{0}^{2} (e^{3x-6} - 1) e^{-3x} \ dx = -\frac{1}{2} e^{6} \sqrt{14} \int_{0}^{2} e^{-6} - e^{-3x} \ dx = -\frac{1}{2} e^{6} \sqrt{14} \left( e^{-6} + \frac{1}{3} e^{-6} - \frac{1}{3} \right) \ dx = \frac{\sqrt{14}}{6} (e^{6} - 7).\]
Let’s extend our derivation of the surface integral of a scalar function to that of a vector field.

The surfaces we consider are “orientable.” A surface $S$ in $\mathbb{R}^3$ is orientable if for any closed curve $C$ on the surface $S$ if a “person” walks the curve, the person ends up on the “same side” as he started.

For instance, a Möbius strip is not orientable. See the picture below.

At ant point $P$ of any orientable surface $S$, there are two possible unit normal vectors; this creates a way to find one set of unit vectors on “one-side” of the surface versus another set of unit vectors on the “other side.” Picking one of these is considered to give an orientation to the surface.
The most common surface integral of a vector field is a “flux” integral. The basic intuition is that flux integral we are about to derive measures “how much” of the vector field passes through the surface.

Consider a vector field $\mathbf{F} = \langle f, g, h \rangle$ in $\mathbb{R}^3$ that is continuous on a region of $\mathbb{R}^3$ and represents a flow (e.g., some fluid). Given a smooth surface $S$ we are interested in how much of $\mathbf{F}$ goes through $S$ (that is, the flux).

We define the flux integral (relative to some orientation of $S$) as

$$\iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

where $d\mathbf{S} = \mathbf{n} \, dS$ and $\mathbf{n}$ is a unit normal to $S$ (at each point on $S$ where it is evaluated).

Suppose we have a parametrization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for $S$ where $(u, v)$ vary over a region $R$ of the $uv$-plane. Then $\mathbf{n} = \pm \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$, so

$$\iiint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \mathbf{F} \cdot \mathbf{n}|\mathbf{t}_u \times \mathbf{t}_v| \, dA = \iint_R \mathbf{F} \cdot (\pm \mathbf{t}_u \times \mathbf{t}_v) \, dA$$

This works as long as $|\mathbf{t}_u \times \mathbf{t}_v| \neq 0$ “almost everywhere,” which means it can be zero at some points, as long as that collection of points forms a subset of $R$ whose area is 0.

The choice of $\pm$ is based on the desired orientation of the surface.

This is also reduced to $\iint_R -f q_z - g q_y + h \, dA$ in the special case of the surface being $z = q(x, y)$ since $\mathbf{r}(x, y) = \langle x, y, q(x, y) \rangle$ implies $\mathbf{t}_x \times \mathbf{t}_y = \langle -q_z, -q_y, 1 \rangle$ (here using the “upward” orientation).
Using the upward orientation, let’s find the flux of the radial vector field \( \mathbf{F} = \langle x, y, z \rangle \) through the surface \( S \) that is the part of the paraboloid \( z = 1 - x^2 - y^2 \) above the \( xy \)-plane. We could use the special case with the graph parametrization, but instead let’s use the following parametrization:

Since \( x^2 + y^2 = 1 - z \) we set \( 1 - z = v^2 \) to get \( \mathbf{r}(u, v) = \langle v \cos(u), v \sin(u), 1 - v^2 \rangle \). Here the region \( R \) in the \( uv \)-plane is determined by \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 1 \).

Then \( \mathbf{t}_u = \langle -v \sin(u), v \cos(u), 0 \rangle \) and \( \mathbf{t}_v = \langle \cos(u), \sin(u), -2v \rangle \) so it follows that \( \mathbf{t}_u \times \mathbf{t}_v = \langle -2v^2 \cos(u), -2v \sin(u), -v \rangle \).

Looking at the third component we see a downward orientation. No problem, just negate.

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \langle v \cos(u), v \sin(u), 1 - v^2 \rangle \cdot \langle 2v^2 \cos(u), 2v^2 \sin(u), v \rangle \, dA
= \iint_R 2v^3 + v - v^3 \, dA
= \int_0^{2\pi} \int_0^1 v^3 + v \, dv \, du
= 2\pi(1/4 + 1/2) = 3\pi/2.
\]
The heat flow vector field for a heat conducting object is $\mathbf{F} = -k \nabla T$ where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material.

Let’s compute the outward flux of the heat flow across the surface $S$ of a spherical object (of radius $R$) where $k = 1$ and $T(x, y, z) = e^{-x^2 - y^2 - z^2}$ is the temperature inside.

We use the parametrization $\mathbf{r}(u, v) = \langle R \sin(u) \cos(v), R \sin(u) \sin(v), R \cos(u) \rangle$, where $R$ is the rectangle $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

We already have $\mathbf{t}_u \times \mathbf{t}_v = R^2 \langle \sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u) \rangle$, which points outward. So since $-\nabla T(x, y, z) = 2T(x, y, z) \langle x, y, z \rangle$ and $T(\mathbf{r}(u, v)) = 2e^{-R^2}$

$$
\iint_S -\nabla T \cdot \mathbf{n} \, dS =
\iint_R 2e^{-R^2} \langle \sin(u) \cos(v), \sin(u) \sin(v), \cos(u) \rangle \cdot R^2 \langle \sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u) \rangle \, dA
= 2R^3 e^{-R^2} \int_0^{2\pi} \int_0^\pi \sin^3(u)(\cos^2(v) + \sin^2(v)) + \sin(u) \cos^2(u) \, du \, dv
= 2R^3 e^{-R^2} (2\pi) \int_0^\pi \sin^3(u) + \sin(u) \cos^2(u) \, du
= 4\pi R^3 e^{-R^2} \int_0^\pi \sin(u) \, du
= 4\pi R^3 e^{-R^2} (-\cos(u) |^\pi_0) = 8\pi R^3 e^{-R^2}.
$$
Consider the radial vector field \( \mathbf{F} = \frac{\mathbf{r}}{||\mathbf{r}||^p} \). Let \( S \) be the sphere of radius \( R \) centered at the origin. Let’s compute the outward flux of \( \mathbf{F} \) across the sphere in two different ways:

First, let’s do it with the parametrization \( s(u, v) = R(\sin(u) \cos(v), \sin(u) \sin(v), \cos(u)) \) where \( R \) is the rectangle \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \).

Recall that for this parametrization, \( \mathbf{t}_u \times \mathbf{t}_v = R^2(\sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u)) \).

\[
\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \\
= \int_R \frac{1}{R^p} R(\sin(u) \cos(v), \sin(u) \sin(v), \cos(u)) \cdot R^2(\sin^2(u) \cos(v), \sin^2(u) \sin(v), \sin(u) \cos(u)) \, dA \\
= \frac{1}{R^{p-3}} \int_0^{2\pi} \int_0^\pi \sin^3(u)(\cos^2(v) + \sin^2(v)) + \sin(u) \cos^2(u) \, du \, dv \\
= \frac{2\pi}{R^{p-3}} \int_0^\pi \sin^3(u) + \sin(u) \cos^2(u) \, du \\
= \frac{2\pi}{R^{p-3}} \int_0^\pi \sin(u) \, du \\
= \frac{4\pi}{R^{p-3}}
\]

Now let’s do it by symmetry and the graph parametrization of \( z = q(x, y) \) for the half of \( x^2 + y^2 + z^2 = R^2 \) that satisfies \( z \geq 0 \). We just double the flux across the top-half of the sphere.

We can find the partials \( z_x \) and \( z_y \) implicitly: 
\[ z_x = -\frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} \] 
and 
\[ z_y = -\frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} \] 

So we can use 
\[ z = \sqrt{R^2 - x^2 - y^2} \] 
and 
\[ \mathbf{n} = \langle -z_x, -z_y, 1 \rangle. \]

\[
\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \\
= 2 \int_{x^2+y^2 \leq R^2} \frac{1}{R^p} \left( \frac{x}{\sqrt{R^2 - x^2 - y^2}}, \frac{y}{\sqrt{R^2 - x^2 - y^2}}, 1 \right) \, dA \\
= 2 \int_{x^2+y^2 \leq R^2} \frac{R^2}{\sqrt{R^2 - x^2 - y^2}} \, dA \\
= 2 \int_{x^2+y^2 \leq R^2} \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA \\
= \frac{4\pi}{R^{p-2}} \left( -\frac{R}{\sqrt{R^2 - R^2}} \right) \\
= \frac{4\pi}{R^{p-2}}
\]
Finally sometimes we use the definition \( \iint_S \mathbf{F} \, dS = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \) directly:

Let’s calculate the (upward) flux integral of the vector fields \( \mathbf{F}_1 = \langle x, 2y, 3z \rangle \) and \( \mathbf{F}_2 = \langle y^2, z^2, x^2 \rangle \) through the disk \( D = \{ (x, y, z) : z = 1, x^2 + y^2 \leq 1 \} \).

We can use \( \mathbf{n} = \langle 0, 0, 1 \rangle \) obviously.

So for \( \mathbf{F}_1 \), we get
\[
\iint_D \mathbf{F}_1 \cdot \mathbf{n} \, dS = \iint_D 3z \, dS = 3 \iint_D 1 \, dS = 3\pi.
\]

And for \( \mathbf{F}_2 \) with the parametrization \( \mathbf{r}(x, y) = \langle x, y, 1 \rangle \) over the unit disk \( D' \) we get (noting that \( dS = dA \)):
\[
\iint_D \mathbf{F}_2 \cdot \mathbf{n} \, dS = \iint_{D'} x^2 \, dA = \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) \, r \, dr \, d\theta = \frac{1}{4} \int_0^{2\pi} (1/2)(1 + \cos(2\theta)) \, d\theta = \frac{\pi}{4}.
\]