14.1 Vector Fields

A function $F$ is a vector field on a subset $S$ of $\mathbb{R}^n$ if $F$ is a function from $S$ to $\mathbb{R}^n$. In particular, this means that

$$F(x_1, x_2, ..., x_n) = \langle f_1(x_1, x_2, ..., x_n), f_2(x_1, x_2, ..., x_n), ..., f_n(x_1, x_2, ..., x_n) \rangle,$$

for some scalar-valued functions $f_1, f_2, ..., f_n$ defined on $S$.

We like to depict a vector field by drawing the outputs as vectors with their “tails” at the input (see below).

Examples:

(1) $F(x, y) = \langle x, y \rangle$ is a vector field on $\mathbb{R}^2$. This is an example of a “radial vector field.” Below is a picture.

Note: $|F(x, y)| = \sqrt{x^2 + y^2}$. So the output vector’s length at an input is equal to the input’s distance to the origin.
(2) \( \mathbf{F}(x, y) = \langle y, -x \rangle \) is a vector field on \( \mathbb{R}^2 \). This is an example of a “rotational vector field.” Below is a picture.

Note: \( |\mathbf{F}(x, y)| = \sqrt{x^2 + y^2} \). So the output vector’s length at an input is equal to the input’s distance to the origin.
(3) $\mathbf{F}(x, y) = \langle y, 0 \rangle$ is a vector field on $\mathbb{R}^2$. This is an example of a “shear vector field.” Below is a picture.

Note: $|\mathbf{F}(x, y)| = |y|$. So the output vector’s length at an input is equal to the input’s distance to the $x$-axis.
(4) $\mathbf{F}(x, y, z) = \left< 0, 0, 1 - \frac{x^2 + y^2}{2} \right>$ is a vector field on $\mathbb{R}^3$. This is an example of a "flow vector field." Below is a picture.
(5) \( \mathbf{F}(x, y) = \left\langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \right\rangle \) is a vector field on \( \{ (x, y) \in \mathbb{R}^2 | y > 0 \} \) (the upper half of \( \mathbb{R}^2 \)).

This is an example of a “gradient vector field.” Below is a picture
Basic Types of Vector Fields:

Let \( \mathbf{r} = (x_1, x_2, \ldots, x_n) \). A radial vector field is one of the form \( \mathbf{F}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)\mathbf{r} \) where \( f \) is a scalar-valued function on \( \mathbb{R}^n \).

We can express such a field in the form

\[
\mathbf{F}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)|\mathbf{r}|\hat{\mathbf{r}},
\]

where \( \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} \) is a unit vector (so \( f(x_1, x_2, \ldots, x_n)|\mathbf{r}| \) is the magnitude).

In particular, we will often use radial vector fields of the form \( \mathbf{F}(x_1, x_2, \ldots, x_n) = \mathbf{r} |\mathbf{r}|^p \) where \( p \) is a real number. See example 1.

A vector field \( \mathbf{F} \) on a subset \( S \) of \( \mathbb{R}^n \) is a gradient vector field if there exists a scalar-valued function \( \phi \) on \( S \) such that \( \mathbf{F} = \nabla \phi \). We call such a function \( \phi \) a potential function. See example 5. Determine a potential function!

Level sets of a potential function \( \phi \) are called equipotential sets. In the case where \( n = 2 \), these are level curves and are called equipotential curves. In the case where \( n = 3 \), these are level surfaces of a potential function \( \phi \) are called equipotential surfaces.

Recall that the gradient evaluated at a point \( P \) is always orthogonal to the level set (curve, surface, etc.) at \( P \) (in the direction of maximum increase). So a gradient vector field is everywhere orthogonal to equipotential sets!

A directed curve through a gradient vector field that is everywhere orthogonal to equipotential sets is a flow curve or streamline.
Consider the vector field \( \mathbf{F}(x, y) = (2x, 1) \). This is a gradient vector field. One potential function is \( \phi(x, y) = x^2 + y \).

Here are some equipotential curves in black and a flow curve in red (shown within the vector field):

Let’s verify that the gradient vector field is orthogonal to its equipotential curves at any point \((x, y)\).

Since the level curve looks like \( \phi(x, y) = k \) for some constant \( k \) we can use the rule \( y' = -\frac{\phi_x}{\phi_y} \) to get \( y' = -\frac{2x}{1} = -2x \). Then \( \mathbf{T}(x, y) = (1, -2x) \) is tangent to the equipotential curve at \((x, y)\). Since \( \mathbf{F}(x, y) = (2x, 1) \) we have that \( \mathbf{F}(x, y) \cdot \mathbf{T}(x, y) = 0 \), which shows \( \mathbf{F} \) is orthogonal to its equipotential curves.
Determine where the gradient vector field $\mathbf{F} = \langle y, x \rangle$ (often out of laziness we write $\mathbf{F}$ for $\mathbf{F}(x, y)$) is tangent to and normal to the curve $C$ parameterized by $\mathbf{r}(t) = \langle t, 1 - t^2 \rangle$.

Let $(x, 1 - x^2)$ be an arbitrary point on the curve. A tangent vector is $\mathbf{T} = \langle 1, -2x \rangle$. Also $\mathbf{F}(x, 1 - x^2) = \langle 1 - x^2, x \rangle$.

- $\mathbf{F}$ is tangent to $C$ wherever $\langle 1 - x^2, x \rangle = c\langle 1, -2x \rangle$ for some non-zero scalar $c$:

  Then $c = 1 - x^2$ and $x = -2cx$. From the second equation, either $x = 0$ or $c = -1/2$.
  
  If $x = 0$ then $c = 1$. If $c = -1/2$ and $x = \pm \sqrt{3}/2$.

  So $\mathbf{F}$ is tangent to curve $C$ at $\left( \pm \frac{\sqrt{3}}{2}, -\frac{1}{2} \right)$ and $(0, 1)$.

- $\mathbf{F}$ is normal to the curve wherever $\mathbf{F}(x, x^2) \cdot \mathbf{T} = 0$:

  So we solve $\langle 1 - x^2, x \rangle \cdot \langle 1, -2x \rangle = 0$, which is equivalent to $1 - x^2 - 2x^2 = 0$.

  Hence $x = \pm \sqrt{1/3}$. So $\mathbf{F}$ is normal to the curve at the $\left( \pm \sqrt{\frac{1}{3}}, \frac{2}{3} \right)$.

The picture confirms our calculations:

![Picture](attachment:image.png)
Now let’s construct a vector field with some desired properties. Let’s find a non-constant vector field on $\mathbb{R}^2$ (excluding the origin) that is always normal to the line $y = x$ and has unit magnitude at every point (excluding the origin).

Let $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$. Since $\mathbf{T} = \langle 1, 1 \rangle$ is tangent to the line $y = x$ we require that $f(x, x) + g(x, x) = 0$. The other requirement is that $f(x, y)^2 + g(x, y)^2 = 1$. There are lots of possibilities, but one is $f(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$ and $g(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$.

Here is this vector field:

The solution above isn’t unique, here is another solution based off of “normalizing” $\mathbf{F} = \langle x - 2y, x \rangle$:
Applications:

- The electric field $\mathbf{E}$ in space due to a point charge at $(0,0,0)$ satisfies $\mathbf{E} = -\nabla V$ where $V(x, y, z) = \frac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $k > 0$ is a constant.

Let's show that the radial component of the electric field is $k/r^2$ where $r = \sqrt{x^2 + y^2 + z^2}$.

$$
\mathbf{E} = \frac{k}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle = \frac{k}{x^2 + y^2 + z^2}\hat{r} \text{ where } \hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle.
$$

Hence the radial component of $\mathbf{E}$ is $k/r^2$ where $r = \sqrt{x^2 + y^2 + z^2}$.

- A flow curve to $\mathbf{F} = (f(x, y), g(x, y))$ satisfies $y' = \frac{g(x, y)}{f(x, y)}$.

Find equations for the flow curves of $\mathbf{F} = (1 + x^2, y^2)$.

$$
y' = \frac{y^2}{1 + x^2} \rightarrow \frac{1}{y^2}y' = \frac{1}{1 + x^2} \rightarrow -\frac{1}{y} = \tan^{-1}(x) - C \rightarrow y = \frac{1}{C - \tan^{-1}(x)}.
$$

Here is the flow curve with $C = 1$: 

![Flow Curve](image)