Chapters 13-14:

(1) If \(2\mathbf{u} = <1, 2, -2>\) then \(|\mathbf{u}|\) is...

(2) Let \(\mathbf{u} = <1, -3, 2>\). Find the values of \(b\) that make \(<1, b, b^2>\) orthogonal to \(\mathbf{u}\).

(3) Describe all vectors orthogonal to both \(<1, -2, 3>\) and \(<-4, -1, 5>\)?

(4) Find a vector equation for the line segment joining the points \((1, 2, 5)\) and \((3, -2, 7)\).

(5) Find the distance from the point \((-7, 2, 3)\) to the line \(\mathbf{r}(t) = <2 + t, -3 + 4t, 8 - 3t>\)?

(6) Find a vector-valued function \(\mathbf{r}(t)\) such that \(\mathbf{r}'(t) = (3\sqrt{t})\mathbf{i} + 2e^{2t}\mathbf{j} + \mathbf{k}\) and \(\mathbf{r}(0) = \mathbf{i} - \mathbf{k}\).

(7) Let \(\mathbf{r}(t) = <\sin t, \cos t, t^2>\). Setup an integral that gives the length of the curve above from \(<0, 1, 0>\) to \(<0, -1, \pi^2>\). DO NOT EVALUATE.

(8) Find the area of the parallelogram with vertices: \((-5, 7), (-3, 12), (-1, 3),\) and \((1, 8)\).

(9) Find a vector valued function for the tangent line at \(t = 1\) for the vector-valued function:

\[
\mathbf{r}(t) = \ln(t + 1)\mathbf{i} + \sin(2\pi t)\mathbf{j} + \mathbf{k}.
\]

(10) Find a vector valued function for the tangent line to the path \(\mathbf{r}(t) = <t \cos t, t \sin t, t>\) at \(t = \frac{\pi}{2}\).

(11) Find the distance from the line \(\mathbf{r}(t) = <-2 + t, 10 - t, -2t>\) to the unit sphere \(x^2 + y^2 + z^2 = 1\).

(12) Derive a formula for the curvature of the curve \(y = f(x)\) where \(f(x)\) is a twice-differentiable function. Find the curvature of the curve \(y = e^{2x}\) at an arbitrary point on the curve as a function of \(x\).
(13) For the curve $\mathbf{r}(t) = < t, -t, t^2 >$, find the curvature as a function of $t$. Where is the curvature the greatest? What is that maximum curvature?

(14) A particle starts at the origin with an initial velocity of $< 1, 0, 4 >$. Its acceleration function is $\mathbf{a}(t) = 3t^2 \mathbf{i} - 12t \mathbf{j} + \mathbf{k}$. Find functions for velocity and position.

(15) Consider the curve of intersection between the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find an integral expression for the arc length along this curve from the origin to $(6, 18, 36)$.

(16) Evaluate the integral:
$$\int_{-1}^{1} \left( \cos(\pi t) \mathbf{i} - 4te^{-t^2} \mathbf{j} + \frac{1}{t^2 + 2t + 2} \mathbf{k} \right) dt.$$

(17) Find the interior angles of the triangle with vertices $(2, 3, 1)$, $(3, 7, 1)$, $(-2, 4, 1)$.

(18) Find the arc length parametrization of the curve $\mathbf{r}(t) = < \sin t, 2 \sin t, \sqrt{5} \cos t >$. Hint: Express $\mathbf{r}(t)$ in terms of $s$ where
$$s(t) = \int_{0}^{t} |\mathbf{r}'(u)| du.$$

(19) Determine if the following curve lies on a sphere:
$$\mathbf{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ -t \end{pmatrix} \left/ \sqrt{1 + t^2} \right.$$

(20) Suppose $\mathbf{r}(t)$ is a path in $\mathbb{R}^n$ and satisfies $|\mathbf{r}(t)| = 1$ for all $t$. Which of the following are true? Select the best answer.
(a) $|\mathbf{r}'(t)| = 0$.
(b) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular.
(c) $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are perpendicular.
(d) All of the above.
Chapter 15:

(1) Find an equation of the plane containing the points \((-7, 0, -2), (-6, 5, 1), \) and \((-9, 1, 0)\).

(2) Find an equation of the plane that contains the point \((1, 8, 6)\) and contains the line \(\ell(t) = <3 + \frac{t}{2}, 2t, 1 - t>\). Find the distance from this plane to the point \((3, -1, 1)\).

(3) Classify which quadratic surface is determined by the equation \(x^2 - 2y^2 - z^2 = 1\). What do the traces look like when \(x\) or \(y\) is set to a constant?

(4) Classify which quadratic surface is determined by the equation \(10x^2 = 5y^2 + 2z\). What do the traces look like when \(x\) or \(y\) is set to a constant?

(5) Plot level curves for \(f(x, y) = x^2y + y\) corresponding to \(f(x, y) = k\) for \(k = -4, -2, 0, 2, 4\).

(6) Let \(\ell\) be the line through the point \((0, 1, 2)\) in the direction of the point \((2, 1, 3)\). Find the intersection of \(\ell\) and the plane \(5x - 2y - z = 7\).

(7) Find the limit or justify that it does not exist:

\[
\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + y^2}.
\]

(8) Find the limit or justify that it does not exist:

\[
\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^2}.
\]

(9) Find all first- and second-order partials of \(V\) as a function of \(r\) and \(h\):

\[
V(r, h) = \frac{\pi h^2}{3}(3r - h)
\]

(10) Find \(f_{xy}\) for the following function:

\[
f(x, y) = x\sqrt{1 + \frac{y}{x}}.
\]
(11) Suppose a cinder cone’s height function (in meters) if given by

\[ h(x, y) = 900 - \sqrt{5x^2 + 4y^2}, \]

where \( x \) and \( y \) represent east-west and north-south respectively; \((0, 0, 900)\) is the top of the cinder cone. Suppose you are on the cinder cone and are 8 meters north of the vertical line through the top of the cinder cone. Find the slope of your path if you hiking in the southwest direction.

(12) If \( z = e^x \sin(xy) \), \( x = st^2 \), and \( y = s/t \), then use the chain rule to find \( z_s \) and \( z_t \) (in terms of \( s, t \)).

(13) Find the directional derivative of \( f(x, y, z) = ze^{x/y^2} \) at the point \((0,1,5)\) in the direction \(<2,1,-2>\).

(14) In which direction (in the \( xy \)-plane) does the function \( f(x, y) = \frac{2x}{y+x^2y} \) increase most rapidly when \( x = 1 \) and \( y = 2 \). What is that maximum rate of increase?

(15) Find an equation of the tangent plane to the ellipsoid \( x^2 + 16y^2 + 4z^2 = 48 \) at the point \((4, -1, 2)\).

(16) Find an equation of the tangent plane to the surface \( 3xyz = z^3 - y + 3x \) at the point \((1,3,3)\).

(17) Find the linear approximation of \( f(x, y) = 3x^2y - 2xy^3 \) at \((2,1)\) and use it to estimate \( f(2.01, 0.99) \).

(18) Find the linear approximation of \( f(x, y, z) = z \tan^{-1} xy \) at \((1,1,2)\).

(19) The volume of a square pyramid of base side length \( s \) and height \( h \) is \( V = \frac{1}{3}s^2h \). Use differentials to approximate the volume increase when the base side length goes from 1 ft to 1.02 ft and the height goes from 3 ft to 3.05 ft.

(20) Assume \( y \) is a positive, differentiable function of \( x \) over the interval \((0,1)\). Furthermore, assume that \( x^4 + y^2 = x^2 \). Find \( \frac{dy}{dx} \).
(21) Which of the following are true about a differentiable function \( f(x, y) \)? Select the best answer.

(a) \( \nabla f \) points in the direction of maximum increase for the function \( f \).
(b) \( |\nabla f| \) is the rate of maximum decrease.
(c) \( \nabla f(x_0, y_0) \) is orthogonal to the level set of \( f \) corresponding to \( f(x_0, y_0) \).
(d) All of these.

(22) Assume an ant is dropped on a hot griddle at the point \((1, 3)\) where the temperature (in \(^\circ\)F) at the point \((x, y)\) is given by \( T(x, y) = 240 − x^2 − 4y + 0.5y^2 \) where \(-8 \leq x \leq 8\) and \(-10 \leq y \leq 10\) represent inches. Assuming the ant travels in the direction that most rapidly decreases the heat, find the path traveled by the ant. Hints: Let \( r(t) = \langle x(t), y(t) \rangle \) represent the path of the ant. We know that \( r(0) = \langle 1, 3 \rangle \) and \( r'(t) \) is in the same direction as \(-\nabla T(r(t))\).

(23) Suppose the directional derivative of \( f(x, y, z) \) at the point \((1, -5, 2)\) in the direction \( \mathbf{u} = \langle 2, 2, 1 \rangle \) is \(-5\) and \( |\nabla f(1, -5, 2)| = 5\). Find \( \nabla f(1, -5, 2) \).

(24) Does there exist a differentiable function \( f(x, y) \) such that \( \nabla f(x, y) = \langle 4xy^3 + \cos(x), 6x^2y^2 + 2y \rangle \)? If so, produce such a function. Otherwise, justify why no such function exists.

(25) Does there exist a differentiable function \( f(x, y) \) such that the gradient satisfies \( \nabla f(x, y) = \langle y \cos(x), y − \sin(x) \rangle \)? If so, produce such a function. Otherwise, justify why no such function exists.

(26) Let \( f(x, y) = e^{-x^2 + 3x} - 3y^2 \). Find and classify all critical points.

(27) Let \( f(x, y) = x^3 − 6xy + 2y^3 \). Find and classify all critical points.
Chapter 16:

(1) Evaluate the iterated integral:
\[ \int_0^1 \int_0^3 (8 - x - 2y)dy\,dx. \]

(2) Evaluate the iterated integral:
\[ \int_{-1}^1 \int_{1}^{2} \frac{1}{y + x^2} dy\,dx. \]

(3) Evaluate the iterated integral:
\[ \int_{-1}^1 \int_{0}^{1-x^2} \frac{xy}{1 + x^2} dy\,dx. \]

(4) Evaluate the iterated integral:
\[ \int_0^1 \int_0^1 y^3 \cos(\pi xy) dy\,dx. \]

(5) Evaluate the double integral below where \( D \) is the triangle determined by \( y \leq x \leq \pi \) and \( 0 \leq y \leq \pi \):
\[ \int \int_D \frac{\sin x}{x} dA. \]

(6) Evaluate the iterated integral:
\[ \int_0^1 \int_y^1 \frac{x}{(1 + xy)^2} dx\,dy. \]

(7) Evaluate the iterated integral:
\[ \int_{-1}^1 \int_{1}^{\sqrt{2-x^2}} \frac{1}{(x^2 + y^2)^{3/2}} dy\,dx. \]

(8) Find the area contained inside the polar curve \( r = 2 - 2 \sin \theta \). Hint: Setup a double integral in polar coordinates that recovers the area of this region.
(9) Evaluate the double integral below where $R$ is the smaller region that is bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$:

$$\int \int_R 5x^2 y\,dA.$$

(10) Evaluate the iterated integral:

$$\int_{-2}^{2} \int_{0}^{2-|x|} xy^2(1 + x^2 y^2)^3 \,dy\,dx.$$

(11) Evaluate the iterated integral:

$$\int_{0}^{1} \int_{\sqrt{x}}^{1} x \sin^{-1} y^5 \,dy\,dx.$$

(12) Evaluate the iterated integral:

$$\int_{0}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} x \,dy\,dx.$$

(13) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + 7y + z = 14$.

(14) Find the volume of the region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 4$.

(15) Find the average value of $f(x, y) = 30xy^2 - 8x^3y$ over the region bounded by $x^2 + y = 8$ and $y = x^2$.

(16) Find the average distance between the points in the disk $D = \{(x, y) | x^2 + y^2 \leq 9\}$ and the origin.

(17) Find the center of mass of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ given the density function $\rho(x, y, z) = 4 - z$.

Hint: Use symmetry to determine two of the coordinates without integration!
(18) Use cylindrical coordinates to find the volume of the region bounded by the plane 
\[ z = 0 \] and the hyperboloid 
\[ z = \sqrt{2} - \sqrt{1 + x^2 + y^2}. \]

(19) Convert the single integral to an iterated double integral, and then evaluate it by 
reversing the order of integration:
\[ \int_0^{1/2} \sin^{-1} (2x) - \sin^{-1} (x) \, dx. \]

(20) Evaluate the iterated integral:
\[ \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2z} \frac{2z}{1 + x^2 + y^2} \, dz \, dy \, dx. \]

(21) Evaluate the triple integral where \( E \) is is given by 
\[ \sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{4 - x^2 - y^2} \] 
(the solid between hemispheres of radius 1 and 2 centered at the origin):
\[ \int \int \int_E \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \, dV. \]

(22) Evaluate the triple integral below where \( E \) is given by \( \sqrt{3} \leq z \leq \sqrt{4 - x^2 - y^2} \):
\[ \int \int \int_E (x^2 + y^2 + z^2) \, dV. \]
KEY:

Ch. 13 – 14 -

(1) \( \frac{3}{2} \).

(2) \( \frac{1}{2} \).

(3) \( \{ < 7, 17, 9 > | t \in \mathbb{R} \} \).

(Answers can vary as long as they describe the same line.)

(4) \( \{ < 1, 2, 5 > + t < 2, -4, 2 > | t \in [0, 1] \} \).

(Answers can vary as long as they describe the same line segment.)

(5) \( \sqrt{105} \).

(6) \[ r(t) = \left( 2t^\frac{3}{2} + 1 \right) \mathbf{i} + (e^{2t} - 1) \mathbf{j} + (t + 1) \mathbf{k} \]

(7) \[ \int_0^\pi \sqrt{1 + 4t^2} \, dt. \]

(8) 28

(9) \[ \ell(t) = \left( \ln(2) + \frac{t}{2} \right) \mathbf{i} + 2\pi t \mathbf{j} + \mathbf{k}. \]

(10) \[ \ell(t) = \left( -\frac{\pi t}{2} \right) \mathbf{i} + \left( \frac{\pi}{2} + t \right) \mathbf{j} + \left( \frac{\pi}{2} + t \right) \mathbf{k}. \]

(11) \( 4\sqrt{5} - 1 \).
(12) \[ \kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \cdot \frac{4e^{2x}}{(1 + 4e^{4x})^{3/2}}. \]

(13) \[ \kappa(t) = \frac{2\sqrt{2}}{(2 + 4t^2)^{3/2}}. \] The curvature reaches a maximum of 1 at \((0, 0, 0)\).

(14) \[ \mathbf{v}(t) = \langle t^3 + 1, -6t^2, t + 4 \rangle \quad \text{and} \quad \mathbf{r}(t) = \left\langle \frac{t^4}{4} + t, -2t^3, \frac{1}{2}t^2 + 4t \right\rangle. \]

(15) \[ \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}} \, dt. \]

(16) \[ \tan^{-1}(2) \kappa. \]

(17) \[ \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}. \]

(18) \[ \mathbf{r}(s) = \left\langle \sin \left( \frac{s}{\sqrt{5}} \right), 2 \sin \left( \frac{s}{\sqrt{5}} \right), \sqrt{5} \cos \left( \frac{s}{\sqrt{5}} \right) \right\rangle \]

(19) This curve lies on the unit sphere as \(|\mathbf{r}(t)| = 1\) for all values of \(t\).

(20) \((b)\)
Ch. 15 -

(1) \[ 7x - 8y + 11z = -71. \]

(2) \[ 36x - y + 16z = 124. \] The distance to the plane from \((3, -1, 1)\) is \(\frac{1}{\sqrt{1553}}\).

(3) It’s a hyperboloid of two sheets. When \(x\) is set to a constant \(k\) such that \(|k| \geq 1\), the trace is either a point or an ellipse. When \(y\) is set to a constant, the trace is a hyperbola.

(4) It’s a paraboloid. When \(x\) or \(y\) is set to a constant, the trace is a parabola.

(5) See the graph below:
(6) \[ \left( \frac{22}{9}, 1, \frac{29}{9} \right). \]

(7) \[ 0. \]

(8) This limit does not exist. Consider the limits under these two trajectories: \( y = 0 \) and \( y = x^3 \).

(9) \[ V_r = \pi h^2, \; V_h = \pi h(2r - h), \; V_{rr} = 0, \; V_{rh} = V_{hr} = 2\pi h, \; \text{and} \; V_{hh} = 2\pi(r - h). \]

(10) \[ f_{xy} = f_{yx} = \frac{y}{4x^2 \left( 1 + \frac{y}{x} \right)^{3/2}}. \]

(11) \[ \sqrt{2}. \]

(12) \[ z_s = e^{st^2}[t^2 \sin(s^2t) + st(t + 1) \cos(s^2t)] \; \text{and} \; z_t = e^{st^2}[2st \sin(s^2t) + s^2 \cos(s^2t)]. \]

(13) \[ \frac{8}{3}. \]

(14) The function increases most rapidly in the direction \(-\mathbf{j}\), and the max. rate of increase is \( \frac{1}{4} \).

(15) \[ x - 4y + 2z = 12. \]

(16) \[ 12x + 5y - 9z = 0. \]

(17) \[ \mathcal{L}(x, y) = 10x - 12; \; f(2.01, 0.99) \approx 8.1. \]

(18) \[ \mathcal{L}(x, y, z) = x + y + \frac{\pi}{4}z - 2. \]

(19) \[ dV = \frac{17}{300}. \; \text{The volume increase is approximately} \; \frac{17}{300} \; \text{cubic feet.} \]
\[
\frac{dy}{dx} = \frac{x(1 - 2x^2)}{y}.
\]

\[(d)\]

The ant will follow the curve \( y = 4 - \frac{1}{\sqrt{x}} \) towards the right, from \( x = 1 \).

\[-\left\langle \frac{10}{3}, \frac{10}{3}, \frac{5}{3} \right\rangle.\]

Yes; \( f(x, y) = 2x^2y^3 + y^2 + \sin x \).

No, the opposite signs cannot be reconciled.

Two critical points: A local maximum at \((1, 0)\) and a saddle point at \((-1, 0)\).

Two critical points: A saddle at \((0, 0)\) and a local minimum at \((\sqrt{4}, \sqrt{2})\).
Ch.16 -

(1) \( \frac{27}{2} \).

(2) \( \frac{\pi \ln (2)}{2} \).

(3) 0.

(4) \( \frac{2}{3\pi^2} \).

(5) 2.

(6) \( \frac{4 - \pi}{4} \).

(7) \( \frac{4 - \pi}{2\sqrt{2}} \).

(8) 6\pi.

(9) 8.

(10) 0.

(11) \( \frac{\pi - 2}{20} \).

(12) \( \frac{16 - 3\pi}{6} \).
(13) \[ \frac{98}{3}. \]

(14) \[ \frac{8\pi(2 - \sqrt{3})}{3}. \]

(15) \[ 0. \]

(16) \[ 2. \]

(17) \[ (0, 0, 2). \]

(18) \[ \frac{\pi(2 - \sqrt{2})}{3}. \]

(19) \[ \int_0^{1/2} \int_0^{2x} \frac{1}{\sqrt{1-y^2}} dy \, dx = \cdots = \frac{\pi + 3 - 3\sqrt{3}}{6}. \]

(20) \[ \pi \ln(5). \]

(21) \[ 2\pi \ln(2). \]

(22) \[ \frac{\pi(128 - 71\sqrt{3})}{10}. \]