Homework 2
Answer Key

1. Let $U, V$ be subspaces of a vector space $W$.

(a) The subspace sum of $U$ and $V$ is $U + V = \{ u + v \mid u \in U; v \in V \}$. Show that $U + V$ is a subspace of $W$.

Solution: We must prove the three subspace axioms:

- **$V$ contains 0:** Since $U$ and $V$ are subspaces, they both contain 0, so $0 = 0 + 0 \in U + V$.

- **$V$ is closed under addition:** Let $u_1 + v_1$ and $u_2 + v_2$ be arbitrary elements of $U + V$ where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then

$$(u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2) \in U + V.$$ 

- **$V$ is closed under scalar multiplication:** Let $u + v$ be an arbitrary element of $U + V$ where $u \in U$ and $v \in V$, and let $\lambda$ be an arbitrary scalar. Then

$$\lambda(u + v) = \lambda u + \lambda v \in U + V$$

(b) Let $x \in W$, but $x \notin V$. Show that for all $v \in V$, $v + x \notin V$.

Solution: We will use proof by contradiction. Suppose that there exists $v \in V$ such that $v + x \in V$. Then

$$x = -v + (v + x) \in V,$$

but this contradicts our assumption that $x \notin V$.

(c) Show that the union $U \cup V$ is a subspace if and only if $U \subseteq V$ or $V \subseteq U$.

Solution: This is an if and only if statement, so we must prove the implication in two directions.

(i) $\left(\Rightarrow\right)$ Assume $U \subseteq V$. Then $U \cup V = V$ which was assumed to be a subspace of $W$. Similarly, if $V \subseteq U$, then $U \cup V = U$ is a subspace of $W$ by assumption.

(ii) $\left(\Leftarrow\right)$ Assume $U \cup V$ is a subspace. For this direction we will use a proof by contradiction. Suppose that $U \nsubseteq V$ and $V \nsubseteq U$. Then there exists $u \in U$ and $v \in V$ such that $u \notin V$ and $v \notin U$. By problem 1(b) we have that

$$u + v \notin U \quad \text{and} \quad u + v \notin V,$$

so $u + v \notin U \cup V$. This contradicts the assumption that $U \cup V$ is a subspace (since subspaces are closed under addition).
(d) The subspace sum $U + V$ is a **direct sum** (denoted $U \oplus V$) if each element in $U + V$ can be written in only one way as a sum $u + v$ where $u \in U$, $v \in V$. Show that $U + V$ is a direct sum if and only if $U \cap V = \{0\}$.

**Solution:** Again, this is an if and only if statement, so we must prove the implication in two directions.

(i) ($\Leftarrow$) Assume $U \cap V = \{0\}$ and let $u_1 + v_1$ and $u_2 + v_2$ be arbitrary elements of $U + V$ where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. We want to show that if $u_1 + v_1$ and $u_2 + v_2$ represent the same element of $U + W$ then $u_1 = u_2$ and $v_1 = v_2$.

Suppose that

$$u_1 + v_1 = u_2 + v_2.$$ 

Rearranging this equation gives

$$u_1 - u_2 = v_2 - v_1.$$ 

Since $u_1 - u_2 \in U$ and $v_2 - v_1 \in V$, both sides of this equation must be in $U \cap V = \{0\}$. Then

$$u_1 - u_2 = 0,$$

$$v_2 - v_1 = 0,$$

so $u_1 = u_2$ and $v_1 = v_2$ as desired.

(ii) ($\Rightarrow$) Assume each element of $U + V$ can be written in only one way as a sum $u + v$ where $u \in U$ and $v \in V$. We will use proof by contradiction for this direction. Suppose that $U \cap V \neq \{0\}$, so there exists a nonzero $w \in U \cap V$. Then there is more than one way to write $0 \in U + V$ as $u + v$. For example, we could let

$$u = v = 0,$$

or we could let

$$u = w$$ and $$v = -w.$$ 

This contradicts our assumption that each element of $U + V$ can be written in only one way as a sum $u + v$ where $u \in U$ and $v \in V$. 

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2. Suppose \( U = \{(x, 2x, y, x + y) \mid x, y \in \mathbb{C}\} \). Show that \( U \) is a subspace of \( \mathbb{C}^4 \) and find another subspace \( V \) such that \( U \oplus V = \mathbb{C}^4 \). Prove your answers!

**Solution:** We want to choose \( V \) so that every element of \( \mathbb{C}^4 \) can be written as \( u + v \) for some \( u \in U \) and \( v \in V \). Notice that in \( U \) we can already get the first and third coordinates to be whatever we want by choosing values for \( x \) and \( y \). However, the second and fourth coordinates are then determined by our choice of \( x \) and \( y \). This means that a good choice for \( V \) is
\[
V = \{(0, a, 0, b) \mid a, b \in \mathbb{C}\}.
\]

Notice that any element \( (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \) can be written as
\[
(x, 2x, y, x + y) + (0, a, 0, b).
\]

Just let
\[
x = z_1, \quad y = z_3, \quad a = z_2 - 2z_1, \quad b = z_4 - (z_1 + z_2).
\]

Therefore \( \mathbb{C}^4 \subseteq U + V \). Since \( U \) and \( V \) are subspaces of \( \mathbb{C}^4 \), we also have that \( U + V \subseteq \mathbb{C}^4 \), so \( U + V = \mathbb{C}^4 \).

We now want to show that this sum is direct. To do this we will use problem 1(d). We need to show that \( U \cap V = \{0\} \). Let
\[
z = (z_1, z_2, z_3, z_4) \in U \cap V.
\]

Since \( z \in V \) we have \( z_1 = z_3 = 0 \). Now consider \( z \) as an element of \( U \). We have
\[
(0, z_2, 0, z_4) = (x, 2x, y, x + y) \quad \text{for some} \quad x, y \in \mathbb{C}.
\]

By the equations in the first and third coordinates, \( x = 0 \) and \( y = 0 \), so
\[
z_2 = 2x = 0 \quad \text{and} \quad z_4 = x + y = 0.
\]

Therefore
\[
z = (0, 0, 0, 0),
\]

so \( U \cap V = \{0\} \). By problem 1(d), \( U + V = \mathbb{C}^4 \) is a direct sum of \( U \) and \( V \), so we can write \( U \oplus V = \mathbb{C}^4 \).

3. Show that \( 1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3 \) forms a basis of \( \mathbb{P}_3(\mathbb{F}) \) (polynomials over the field \( \mathbb{F} \) of degree at most 3).

**Solution:** Let
\[
B = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}.
\]

To show that \( B \) is a basis for \( \mathbb{P}_3(\mathbb{F}) \) we need to show that it spans \( \mathbb{P}_3(\mathbb{F}) \) and is linearly independent.

Notice that
\[
1 = 1,
\]
\[
x = (1 + x) - (1),
\]
\[
x^2 = (1 + x + x^2) - (1 + x),
\]
\[
x^3 = (1 + x + x^2 + x^3) - (1 + x + x^2),
\]

so \( 1, x, x^2, x^3 \in \text{span}(B) \). Since \( \{1, x, x^2, x^3\} \) is a basis for \( \mathbb{P}_3(\mathbb{F}) \) it spans all of \( \mathbb{P}_3(\mathbb{F}) \), so we must also have that \( B \) also spans all of \( \mathbb{P}_3(\mathbb{F}) \).

To see that \( B \) is linearly independent, notice that \( B \) is a set of four vectors that spans the 4-dimensional space \( \mathbb{P}_3(\mathbb{F}) \). Therefore \( B \) is a basis for \( \mathbb{P}_3(\mathbb{F}) \), so \( B \) is linearly independent.

The fact that a set of \( n \) vectors spanning an \( n \)-dimensional space is a basis was covered on recitation worksheet 3 and is also covered in the book *Linear Algebra Done Right*. 

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4. Let $U, V$ be vector spaces and $T: U \to V$ be a linear transformation.

(a) Show that if $v_1, v_2, \ldots, v_k \in U$ and if $Tv_1, Tv_2, \ldots, Tv_k$ are linearly independent then $v_1, v_2, \ldots, v_k$ are linearly independent.

**Solution:** Suppose $v_1, v_2, \ldots, v_k \in U$ and that $Tv_1, Tv_2, \ldots, Tv_k$ are linearly independent. Set a linear combination of the $v_i$ equal to zero:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \quad \text{for scalars } c_1, c_2, \ldots, c_k.$$  

We want to show that $c_i = 0$ for all $i = 1, \ldots, k$. Apply the transformation $T$ to both sides of the equation and use linearity:

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = T(0)$$

$$\downarrow$$

$$c_1Tv_1 + c_2Tv_2 + \cdots + c_kTv_k = 0$$

Since $Tv_1, Tv_2, \ldots, Tv_k$ are linearly independent, we must have that $c_i = 0$ for all $i = 1, \ldots, k$.

(b) $T$ is injective if for every $u, v \in U$, $Tu = Tv$ implies $u = v$. An injective linear transformation is called a monomorphism. Show that $T$ is a monomorphism if and only if the null space (aka kernel) of $T$ is $\{0\}$.

**Solution:** we must prove the implication in both directions.

(i) $(\Rightarrow)$ Assume $T$ is a monomorphism. Let $u \in \text{null}(T)$. Then by the definition of the null space, $T(u) = 0$. Therefore

$$Tu = 0 = T(0).$$

Since $T$ is a monomorphism (i.e., is injective) we must have $u = 0$, so $\text{null}(T) = \{0\}$.

(ii) $(\Leftarrow)$ Assume $\text{null}(T) = \{0\}$. Let $u, v \in U$ such that $Tu = Tv$. Then

$$T(u - v) = Tu - Tv$$

$$= 0,$$

so $u - v \in \text{null}(T)$. Since $\text{null}(T) = \{0\}$, we have $u - v = 0$, so $u = v$ as desired.

(c) Show that if $T$ is a monomorphism and $v_1, v_2, \ldots, v_k \in U$ are linearly independent then $Tv_1, Tv_2, \ldots, Tv_k$ are linearly independent.

**Solution:** Assume $T$ is a monomorphism and suppose $v_1, v_2, \ldots, v_k \in U$ are linearly independent. Set a linear combination of the $Tv_i$ equal to zero:

$$c_1Tv_1 + c_2Tv_2 + \cdots + c_kTv_k = 0 \quad \text{for scalars } c_1, c_2, \ldots, c_k.$$  

We want to show that $c_i = 0$ for all $i = 1, \ldots, k$. Using linearity, we can rewrite the above equation as

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = T(0).$$

Since $T$ is injective, we have

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$  

Finally, since $v_1, \ldots, v_k$ are linearly independent we have $c_i = 0$ for all $i = 1, \ldots, k$.  

5. Let \( V \) be a vector space over a field \( \mathbb{F} \) and \( T \in \mathcal{L}(V, \mathbb{F}) \). Suppose \( v \in V \) is not in \( \ker(T) \).

**Solution:** Since \( \ker(T) \) and \( \{cv \mid c \in \mathbb{F}\} \) are subsets of \( V \), their sum is also a subset of \( V \).

We now want to show that \( V \subseteq \ker(T) + \{cv \mid c \in \mathbb{F}\} \).

Let \( u \) be an arbitrary element of \( V \), and consider the vector

\[
w = u - \frac{T}{Tv} v \in V
\]

(note that \( Tv \neq 0 \) since \( v \notin \ker(T) \)). If we apply \( T \) to \( w \) we get

\[
T(w) = T \left( u - \frac{T}{Tv} v \right)
\]

\[
= Tu - \frac{T}{Tv} T v
\]

\[
= Tu - Tu
\]

\[
= 0,
\]

so \( w \in \ker(T) \). Therefore

\[
u = w - \frac{T}{Tv} v \in \ker(T) + \{cv \mid c \in \mathbb{F}\},
\]

so \( V \subseteq \ker(T) + \{cv \mid c \in \mathbb{F}\} \) and hence \( V = \ker(T) + \{cv \mid c \in \mathbb{F}\} \).

We now want to show that this sum is direct. Let \( u \in \ker(T) \cap \{cv \mid c \in \mathbb{F}\} \). Since \( u \in \{cv \mid c \in \mathbb{F}\} \) we can write

\[
u = \lambda v \quad \text{for some } \lambda \in \mathbb{F}.
\]

Since \( u \in \ker(T) \) we have

\[
0 = Tu
\]

\[
= T(\lambda v)
\]

\[
= \lambda Tv.
\]

\( v \) was assumed to not be in \( \ker(T) \), so \( T(v) \neq 0 \). Thus we can divide both sides by \( Tv \) to get \( \lambda = 0 \). Therefore

\[
u = 0v = 0,
\]

so \( \ker(T) \cap \{cv \mid c \in \mathbb{F}\} = \{0\} \). By problem 1(d) the subspace sum is direct, and we can write

\[
V = \ker(T) \oplus \{cv \mid c \in \mathbb{F}\}.
\]