Homework 3
Answer Key

1. Let \( G: P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) be given by \( G(f(x)) = (x + 1)f'(x) - 2f(x) \).

(a) Show that \( G \) is a linear map.

Solution: We must show the two properties of a linear map:

- **Additivity:** Let \( u(x), v(x) \in P_2(\mathbb{R}) \). Then

  \[
  G(u(x) + v(x)) = (x + 1)(u(x) + v(x))' - 2(u(x) + v(x)) \\
  = (x + 1)(u'(x) + v'(x)) - 2u(x) - 2v(x) \\
  = [(x + 1)u'(x) - 2u(x)] + [(x + 1)v'(x) - 2v(x)] \\
  = G(u(x)) + G(v(x)).
  \]

- **Homogeneity:** Let \( f(x) \in P_2(\mathbb{R}) \) and \( \lambda \in \mathbb{R} \). Then

  \[
  G(\lambda f(x)) = (x + 1)(\lambda f(x))' - 2(\lambda f(x)) \\
  = (x + 1)\lambda f'(x) - 2\lambda f(x) \\
  = \lambda [(x + 1)f'(x) - 2f(x)] \\
  = \lambda G(f(x)).
  \]

(b) Find a basis for \( \ker(G) \) (aka \( \text{null}(G) \)). What is the dimension of \( \ker(G) \)?

Solution: Let \( f(x) \in \ker(G) \). Since \( f(x) \in P_2(\mathbb{R}) \) we can write

\[
 f(x) = ax^2 + bx + c \quad \text{for some } a, b, c \in \mathbb{R}.
\]

Now since \( f(x) \) is in \( \ker(G) \) we have \( G(f(x)) = 0 \), so

\[
 0 = G(f(x)) = G(ax^2 + bx + c) = (x + 1)(2ax + b) - 2(ax^2 + bx + c) = (2a - b)x + (b - 2c).
\]

Since \( x \) and 1 are linearly independent, the only way to get \( 0 = (2a - b)x + (b - 2c) \) is if

\[
 2a - b = 0 \quad \text{and} \quad b - 2c = 0.
\]

Therefore \( 2a = b = 2c \), so \( f(x) \) can be written as

\[
 f(x) = ax^2 + 2ax + a = a(x^2 + 2x + 1).
\]

Since \( f(x) \) was an arbitrary element of \( \ker(G) \), we can write every element of \( \ker(G) \) as \( a(x^2 + 2x + 1) \) for some \( a \in \mathbb{R} \). That is,

\[
 \ker(G) = \{a(x^2 + 2x + 1) : a \in \mathbb{R}\} = \text{span}\{x^2 + 2x + 1\}
\]

Now \( \{x^2 + 2x + 1\} \) has only one (nonzero) element, so it is linearly independent. It is therefore a basis for \( \ker(G) \). The dimension of \( \ker(G) \) is

\[
 \text{nullity}(G) = \dim(\ker(G)) = 1.
\]
(c) Find a basis of $G(P_2(\mathbb{R}))$ (the range of $G$). What is the rank of $G$?

**Solution:** The range of $G$ is the set of all outputs of the map $G$. That is

$$\text{range}(G) = \{G(f(x)) : f(x) \in P_2(\mathbb{R})\}.$$ 

If $f(x) \in P_2(\mathbb{R})$ we can write

$$f(x) = ax^2 + bx + c, \quad \text{for some } a, b, c \in \mathbb{R},$$

so

$$\text{range}(G) = \{G(ax^2 + bx + c) : a, b, c \in \mathbb{R}\} = \{(2a - b)x + (b - 2c) : a, b, c \in \mathbb{R}\}.$$ 

Since $(2a - b)x + (b - 2c)$ is a first-degree polynomial, it must be contained in the span of $\{x, 1\}$. That is,

$$\text{range}(G) \subseteq \text{span}(\{x, 1\})$$

Now notice that letting $a = b = c = 1$ gives

$$(2a - b)x + (b - 2c) = x,$$

so $x \in \text{range}(G)$. Similarly, letting $a = 1$, $b = 2$, and $c = \frac{1}{2}$ gives

$$(2a - b)x + (b - 2c) = 1,$$

so $1 \in \text{range}(G)$. Therefore $\text{span}(\{x, 1\}) \subseteq \text{range}(G)$, so

$$\text{range}(G) = \text{span}(\{x, 1\}).$$

Since $\{x, 1\}$ is linearly independent, it is a basis for $\text{range}(G)$. The rank of $G$ is

$$\text{rank}(G) = \dim(\text{range}(G)) = 2.$$ 

(d) Is $G$ a monomorphism, epimorphism (an onto linear map), and/or an isomorphism?

**Solution:**

- We showed in homework 2 that $G$ is a monomorphism if and only if $\ker(G) = \{0\}$. Since this is not true by part (b) above, $G$ is not a monomorphism.

- To be an epimorphism we would need the range of $G$ to be equal to the codomain. That is, we would need $\text{range}(G) = P_2(\mathbb{R})$. Since this is not true by part (c) above, $G$ is not an epimorphism.

- To be an isomorphism, $G$ needs to be both a monomorphism and an epimorphism. Since it is neither a monomorphism nor an epimorphism, $G$ is not an isomorphism.
2. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2 \times 2}(\mathbb{C}) : a + b + c + id = 0 \right\}$ and let $H : V \to P_2(\mathbb{C})$ be given by

$$H \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b)z^2 + (b + c)z + (c + d),$$

which is linear.

(a) Show that $V$ is a subspace of $M_{2 \times 2}(\mathbb{C})$.

**Solution:** A valid way to solve this problem is to prove the three subspace axioms:

- $V$ is closed under addition,
- $V$ is closed under scalar multiplication,
- $V$ contains the zero vector.

However, I am going to use a different argument. I will show that $V$ is the span of a set of vectors. The span of any set of vectors is always a vector space.

Let $A$ be an arbitrary element of $V$, so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}$$

with

$$a + b + c + id = 0.$$  

From this condition we can solve for $a$:

$$a = -b - c - id,$$

so we can write $A$ as

$$A = \begin{bmatrix} -b - c - id & b \\ c & d \end{bmatrix}, \quad b, c, d \in \mathbb{C}$$

Since $A$ was an arbitrary element of $V$, we can rewrite $V$ as

$$\left\{ \begin{bmatrix} -b - c - id & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{C} \right\} = \left\{ b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} : b, c, d \in \mathbb{C} \right\}$$

Notice that

$$b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

is a linear combination of three matrices, so $V$ is the set of all linear combinations of these three matrices. That is,

$$V = \text{span} \left( \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\} \right),$$

so $V$ is a vector space.
(b) Find a basis of $V$.

**Solution:** From our work in part (a) we found that $V = \text{span}(B)$ where

$$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$ 

We now want to show that $B$ is linearly independent. Set a linear combination of the vectors equal to 0:

$$b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} -b - c - id \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives $b = 0$, $c = 0$, and $d = 0$. Therefore $B$ is linearly independent, so it is a basis for $V$.

(c) Find a matrix representation of $H$. Clearly identify the bases that you are using.

**Solution:** For $V$ we will use the basis

$$B_1 = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

that we found in part (b). For $P_2(\mathbb{C})$ we will use the basis $B_2 = \{z^2, z, 1\}$. Recall that we find the columns of the matrix for $H$ by applying $H$ to the elements of $B_1$ and finding the coordinates with respect to $B_2$.

$$H \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) = (1 - 1)z^2 + (-1, 0)z + (0 + 0) = (0)z^2 + (-1)z + (0)1$$

so the first column of our matrix is

$$\left[ H \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \right]_{B_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$ 

The other calculations are

$$H \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = (-1)z^2 + (1)z + (1)1$$

and

$$H \left( \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \right) = (-i)z^2 + (0)z + (1)1,$$

so the matrix for $H$ is

$$[H]_{B_2,B_1} = \begin{bmatrix} 0 & -1 & -i \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
(d) Find the dimension of \( \ker(H) \).

**Solution:** To find the dimension of the kernel, we can row-reduce the matrix found in part (c). The number of non-pivot columns will be the dimension of \( \ker(H) \).

\[
\begin{bmatrix}
0 & -1 & -i \\
-1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

\[
\xrightarrow{\text{RREF (details omitted)}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Since there are no non-pivot columns in the reduced matrix, we get

\[
\text{nullity}(H) = \dim(\ker(H)) = 0.
\]

(e) Find the rank of \( H \).

**Solution:** The rank-nullity theorem tells us

\[
\text{rank}(H) + \text{nullity}(H) = \dim(V)
\]

\[
\downarrow
\]

\[
\text{rank}(H) + 0 = 3
\]

\[
\downarrow
\]

\[
\text{rank}(H) = 3.
\]
3. Let $V$ be the subspace of $M_{2 \times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each row is equal to 0. Let $W$ be the subspace of $M_{2 \times 2}(\mathbb{R})$ consisting of all matrices in which the sum of entries on each column is equal to 0. Find a basis of $V + W$.

**Solution:** We first want to find bases for $V$ and $W$. Notice that we can write $V$ as

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2 \times 2}(\mathbb{C}) : a + b = 0, c + d = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2 \times 2}(\mathbb{C}) : b = -a, d = -c \right\}$$

$$= \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} : a, c \in \mathbb{C} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, c \in \mathbb{C} \right\} ,$$

so $V = \text{span}(B_1)$ where

$$B_1 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\} .$$

Since one matrix in $B_1$ is not a scalar multiple of the other, $B_1$ is linearly independent, so it is a basis for $V$. By very similar arguments we get the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

for $W$.

Now consider the set

$$B_1 \cup B_2 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\} .$$

This is not necessarily a basis for $V + W$, but we can use it to find a basis.

Using the standard basis for $M_{2 \times 2}(\mathbb{C})$ we can represent a $2 \times 2$ matrix by a $4 \times 1$ column vector:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Write each of the matrices in $B_1 \cup B_2$ as $4 \times 1$ column vectors in this way, and let them form the columns of a matrix; then apply row-reduction:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF (details omitted)}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The first three columns of the reduced matrix are pivot columns, while the fourth column is not a pivot column. Therefore, the first three elements of $B_1 \cup B_2$ form a basis for $V + W$. That is,

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right\}$$

is a basis for $V + W$. 
4. Let $V$ be a vector space with basis $\mathbf{B}_1 = \{v_1, v_2, \ldots, v_7\}$, and let $W$ be a vector space with basis $\mathbf{B}_2 = \{w_1, w_2, \ldots, w_6\}$. Let $f: V \to W$ be the linear map determined by

\[
\begin{align*}
  f(v_1) &= w_1 + w_2 - w_4 + 2w_6, \\
  f(v_2) &= 3w_1 - w_2 - w_3 + w_5 - 4w_6, \\
  f(v_3) &= 2w_2 + 5w_3 - w_4 + 7w_5 - w_6, \\
  f(v_4) &= w_1 + w_3 - w_4 + w_6, \\
  f(v_5) &= w_2 - 4w_4 + 5w_5 + 3w_6, \\
  f(v_6) &= w_1 + w_2 + 2w_3 + 3w_4 + 5w_5, \\
  f(v_7) &= 2w_1 - 6w_3 + 2w_4 + w_5 - w_6.
\end{align*}
\]

(a) Write the matrix that represents $f$ relative to bases $\mathbf{B}_1$ and $\mathbf{B}_2$.

**Solution:** Remember that each $f(v_i)$ gives a column of the matrix. The entries in the matrix are the coefficients of the $w_j$ vectors:

\[
[f]_{\mathbf{B}_2, \mathbf{B}_1} = \begin{bmatrix}
  1 & 3 & 0 & 1 & 0 & 1 & 2 \\
  1 & -1 & 2 & 0 & 1 & 1 & 0 \\
  0 & -1 & 5 & 1 & 0 & 2 & -6 \\
  -1 & 0 & -1 & -1 & -4 & 3 & 2 \\
  0 & 1 & 7 & 0 & 5 & 5 & 1 \\
  2 & -4 & -1 & 1 & 3 & 0 & -1
\end{bmatrix}.
\]

(b) Find the rank and nullity of $f$.

**Solution:** We can find the rank of $f$ by row-reducing the matrix $[f]_{\mathbf{B}_2, \mathbf{B}_1}$. The rank will be the number of pivot columns in the reduced matrix. It would be too difficult to reduce the matrix by hand, so we will use MATLAB:

\[
A = \begin{bmatrix}
  1 & 3 & 0 & 1 & 0 & 1 & 2 \\
  1 & -1 & 2 & 0 & 1 & 1 & 0 \\
  0 & -1 & 5 & 1 & 0 & 2 & -6 \\
  -1 & 0 & -1 & -1 & -4 & 3 & 2 \\
  0 & 1 & 7 & 0 & 5 & 5 & 1 \\
  2 & -4 & -1 & 1 & 3 & 0 & -1
\end{bmatrix};
\]

\[
\text{rref}(A)
\]

You should get

\[
\begin{bmatrix}
  1.0000 & 0 & 0 & 0 & 0 & 0 & 1.2913 \\
  0 & 1.0000 & 0 & 0 & 0 & 0 & 0.8434 \\
  0 & 0 & 1.0000 & 0 & 0 & 0 & -0.7987 \\
  0 & 0 & 0 & 1.0000 & 0 & 0 & -2.4803 \\
  0 & 0 & 0 & 0 & 1.0000 & 0 & 0.4909 \\
  0 & 0 & 0 & 0 & 0 & 1.0000 & 0.6586
\end{bmatrix}
\]

Since there are six pivot columns in the reduced matrix, $f$ has rank 6. We can then get the nullity from the rank-nullity theorem:

\[
\text{rank}(f) + \text{nullity}(f) = \dim(V)
\]

\[
\downarrow
\]

\[
6 + \text{nullity}(f) = 7
\]

\[
\downarrow
\]

\[
\text{nullity}(f) = 1.
\]