Instructions: Put your solutions on a separate piece(s) of paper. You may use this page as a cover sheet. Make sure to write your arguments coherently in full sentences. Start a sentence with words. Use words to transition, for example “This leads to”, “Therefore”, etc.

Note: For vector spaces $V, W$ and $S, T \in \mathcal{L}(V, W)$ the expression $ST$ denotes $S \circ T$.

(1) Consider $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ given by $T(f(x)) = x \frac{d}{dx} (f(2x + 1))$.
   (a) Find $[T]_{B}$, the matrix of $T$ with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{P}_2$.
   (b) Find the eigenvalues of $T$.
   (c) For each eigenvalue, find the eigenvectors associated to the eigenvalue (as polynomials in $\mathbb{P}_2$).
   (d) Find a basis $B'$ of $\mathbb{P}_2$ such that $[T]_{B'}$ is a diagonal matrix (if possible).

(2) Let $V$ be real vector space (possibly infinite-dimensional), $S, T \in \mathcal{L}(V)$, and $S$ be invertible. Prove $\lambda \in \mathbb{C}$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue of $STS^{-1}$. Give a description of the set of eigenvectors of $STS^{-1}$ associated to an eigenvalue $\lambda$ in terms of the eigenvectors of $T$ associated to $\lambda$.

(3) Show that there exist square matrices $A, B$ that have the same eigenvalues, but aren’t similar. Hint: Use the identity matrix as one of the matrices.

(4) Prove that $\text{trace}(AB) = \text{trace}(BA)$ for any $A, B \in M_{nn}(\mathbb{C})$, and as a consequence, that similar matrices have the same trace.

   Hint: The $(i, i)$ entry of $AB$ is $\sum_k a_{ik}b_{ki}$.

   BONUS EXERCISE: You are encouraged to use computer software (Matlab, Mathematica, etc.) – Turn in your code with the solution.

(5) Let $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$.
   (a) Find the eigenvalues of $P$.
   (b) For each eigenvalue $\lambda$ of $P$, find a basis of the eigenspace $E_\lambda = \text{Null}(P - \lambda I_4)$.
   (c) Is $P$ diagonalizable? If so, find a diagonal matrix $D$ and an invertible matrix $S$ such that $P = SDS^{-1}$. 

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