Note: A random subset of these problems will be graded for credit.

(1) Determine whether each operation is indeed a binary operation on the indicated set and if so, determine if it is commutative and/or associative.
   
   (a) \(\ast\) on \(\mathbb{Z}^+ = \{a \in \mathbb{Z} | a > 0\}\) given by \(a \ast b = ab\).
   
   (b) \(\ast\) on \(\mathbb{R}^+ = \{a \in \mathbb{R} | a > 0\}\) given by \(a \ast b = a/b\).
   
   (c) \(\ast\) on \(\mathbb{Q}\) given by \(a \ast b = ab\).
   
   (d) \(\ast\) on \(\mathbb{R}\) given by \(a \ast b = ab + a + b\).
   
   (e) \(\ast\) on \(M_n(\mathbb{R})\) (\(n \times n\) matrices with real coefficients) given by \(A \ast B = B^T A\).
   
   (f) \(\ast\) on \(\mathbb{C}\) given by \(z_1 \ast z_2 = z_1 + iz_2\).
   
   (g) \(\ast\) on \(S = \{f : \mathbb{R} \to \mathbb{R}\}\) given by \(f \ast g = f \circ g\) (function composition).

(2) Let \(S\) be a set with \(n\) elements. How many binary operations are there on \(S\)? Justify your answer!

(3) Suppose \(\ast\) is an associative and commutative binary operation on a set \(S\) and there is an identity element \(e\). Show that the set of idempotents \(I = \{a \in S | a \ast a = a\}\) is closed under \(\ast\).

(4) The function \(\phi : \mathbb{Q} \to \mathbb{Q}\) defined by \(\phi(x) = 2x + 3\) is a bijection. Give a binary operation \(\ast\) on \(\mathbb{Q}\) such that \(\phi\) is an isomorphism of \(\mathbb{Q}\) as a binary structure under the usual addition \(+\) to \(\mathbb{Q}\) as a binary structure under \(\ast\). Then give the identity element for \(\ast\).

(5) Let \(G\) be a finite group with an even number of elements. Prove that there exists a non-identity element \(a \in G\) that is its own inverse (that is, \(a^{-1} = a\)).

   Hint: Consider the set \(S = \{a \in G | a' \neq a\}\) and argue that \(S\) must have an even number of elements. Then think about \(G \setminus S\).

(6) Let \(G\) be a group with identity \(e\). Show that if \(x \ast x = e\) for all \(x \in G\) then \(G\) is abelian.

(7) Let \(G\) be a group and \(g \in G\). Show that \(i_g : G \to G\) given by \(i_g(x) = gxg^{-1}\) is a group isomorphism of \(G\) to itself (such an isomorphism is called an automorphism).

(8) If \(G\) is a finite group with identity \(e\), then show that for any \(g \in G\), there exists \(n \in \mathbb{N}\) such that \(g^n = e\). Furthermore, show that \(n\) is less than or equal to the order of \(G\).