Math 343 - Intro. to Modern Algebra **Homework 1** **Due: Jan. 18th, 2019**

Note: A random subset of these problems will be graded for credit.

(1) Determine whether each operation is indeed a binary operation on the indicated set and if so, determine if it is commutative and/or associative.

(a) $*$ on $\mathbb{Z}^+ = \{a \in \mathbb{Z} | a > 0\}$ given by $a * b = ab$.

(b) $*$ on $\mathbb{R}^+ = \{a \in \mathbb{R} | a > 0\}$ given by $a * b = a/b$.

(c) $*$ on $\mathbb{Q}$ given by $a * b = a^b$.

(d) $*$ on $\mathbb{R}$ given by $a * b = ab + a + b$.

(e) $*$ on $M_n(\mathbb{R})$ ($n \times n$ matrices with real coefficients) given by $A * B = B^T A$.

(f) $*$ on $\mathbb{C}$ given by $z_1 * z_2 = z_1 + iz_2$.

(g) $*$ on $S = \{f : \mathbb{R} \to \mathbb{R}\}$ given by $f * g = f \circ g$ (function composition).

(2) Let S be a set with n elements. How many binary operations are there on S? Justify your answer!

(3) Suppose $*$ is an associative and commutative binary operation on a set S and there is an identity element e. Show that the set of *idempotents* $I = \{a \in S | a * a = a\}$ is closed under $*$.

(4) The function $\phi : \mathbb{Q} \to \mathbb{Q}$ defined by $\phi(x) = 2x + 3$ is a bijection. Give a binary operation $*$ on $\mathbb{Q}$ such that ϕ is an isomorphism of $\mathbb{Q}$ as a binary structure under the usual addition $+$ to $\mathbb{Q}$ as a binary structure under $*$. Then give the identity element for $*$.

(5) Let G be a finite group with an even number of elements. Prove that there exists a non-identity element $a \in G$ that is its own inverse (that is, $a^{-1} = a$).

Hint: Consider the set $S = \{a \in G | a' \neq a\}$ and argue that S must have an even number of elements. Then think about $G \setminus S$.

(6) Let G be a group with identity e. Show that if $x * x = e$ for all $x \in G$ then G is abelian.

(7) Let G be a group and $g \in G$. Show that $i_g : G \to G$ given by $i_g(x) = gxg^{-1}$ is a group isomorphism of G to itself (such an isomorphism is called an *automorphism*).

(8) If G is a finite group with identity e, then show that for any $g \in G$, there exists $n \in \mathbb{N}$ such that $g^n = e$. Furthermore, show that n is less than or equal to the order of G.