(1) Suppose $G$ is a finite binary structure under an associative product, where there is both left and right cancelation. Prove $G$ is a group.

(2) Suppose $G$ is a group. Prove that no matter how you bracket the product of any number of group elements, the result is the same (e.g., $(ab)(cd) = a(b(cd))$ for all $a, b, c, d \in G$).

Hint: Use induction (on the number of factors).

(3) Give an example of a group $G$ for which $H = \{ a \in G | a^2 = e \}$ is NOT a subgroup.

(4) Suppose $H, K \leq G$ and that the sets $KH = \{ kh | k \in K, h \in H \}$ and $HK = \{ hk | h \in H, k \in K \}$ are equal. Prove $HK \leq G$.

(5) Give an example of a group $G$ and subgroups $H, K$ such that $HK$ is NOT a subgroup.

(6) Let $G$ be a group and $H \leq G$. For $x \in G$ we have $Hx = \{ hx | h \in H \}$ (called a right-coset of $H$ in $G$). Show that for all $a, b \in G$ either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

(7) For each set of matrices, determine if it is a group (under matrix multiplication — which is associative). Justify your answers.

(a) Invertible $2 \times 2$ matrices with real coefficients with 0 below the main diagonal.
(b) Invertible $3 \times 3$ matrices with rational coefficients such that the main diagonal is all 0s.
(c) Invertible $3 \times 3$ matrices with rational coefficients such that the main diagonal is all 1s.
(d) $n \times n$ matrices with complex coefficients and determinant equal to 1, where $n$ is any positive integer.

(8) Let $G$ be a group and $g \in G$ be a fixed element. The centralizer of $g$ in $G$ is $C_G(g) = \{ a \in G | ag = ga \}$. Show that $C_G(g) \leq G$.

(9) Let $G$ be a group. Prove that if $(ab)^2 = a^2b^2$ for all $a, b \in G$ then $G$ is abelian.