Math 343 - Intro. to Modern Algebra **Homework 3** **Due: Feb. 8th, 2019**

Note: A random subset of these problems will be graded for credit.

(1) Show any group of order 4 is abelian.

Hint: We know cyclic groups are abelian.

(2) Let G, H be groups. Prove that if $\phi : G \to H$ is a homomorphism then the image of G under ϕ, that is $\phi(G) = \{\phi(x) | x \in G\}$, is a subgroup of H.

(3) Suppose G is a group with normal subgroups M, N such that $M \cap N = \{e\}$ where e is the identity of G. Show that for any $m \in M$ and for any $n \in N$, $mn = nm$.

(4) Let G be a group and $Z(G) = \{z \in G | zg = gz \ \forall g \in G\}$ (called the *center* of G). Show that $Z(G) \trianglelefteq G$ and if $G/Z(G)$ is a cyclic group then G is abelian.

(5) Consider the additive quotient group $G = \mathbb{Q}/\mathbb{Z}$.

(a) Show every element of G can be represented by $r + \mathbb{Z}$ where $r \in \mathbb{Q}$ and $0 \leq r < 1$.

(b) Show that every element of G has finite order, but there is no upper bound on the orders of all the elements.

(6) Let G, G' be groups with normal subgroups H, H' respectively. Show that if $\phi : G \to G'$ is a homomorphism and $\phi(H) \subseteq H'$ then $\phi_* : G/H \to G'/H'$ given by $\phi_*(gH) = \phi(g)H'$ is a homomorphism.

(7) We know that $\mathbb{C}^*$, $\mathbb{R}^+$ and $U = \{z \in \mathbb{C} : |z| = 1\}$ are groups under multiplication. Prove that $\mathbb{C}^*/U \cong \mathbb{R}^+$.

Hint: Use the First Isomorphism Theorem!

(8) Let G be a group and $H \leq G$. The *normalizer of H in G* is

$$N(H) = \{g \in G | gHg^{-1} = H\}.$$

Prove the following:

(a) $N(H) \leq G$.

(b) H is a normal subgroup of $N(H)$.

Hint: First show that $H \subseteq N(H)$.

(c) If H is a normal subgroup of $K \leq G$ then $K \leq N(H)$. Conclude that $N(H)$ is the largest subgroup of G in which H is normal.

(d) H is a normal subgroup of G iff $G = N(H)$.

One presentation of the Dihedral Group of order 8 (symmetries of a square) is $D_8 = \{\mathrm{Id}, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$ with the relations $\sigma^4 = \mathrm{Id}$, $\tau^2 = \mathrm{Id}$ and $\sigma\tau = \tau\sigma^3$.

(9) Consider the subgroup $H = \langle\tau\rangle$ of D_8. Determine $N(H)$, the normalizer of H in D_8.

(10) Let G be a group and $a, b \in G$ and assume b is **NOT** the identity element.

(a) Show that if $aba^{-1} = b^i$ for some $i \in \mathbb{N}$ then $a^r b a^{-r} = b^{i^r}$ for all $r \in \mathbb{N}$.

Hint: Use induction!

(b) If the order of a is 5 and $aba^{-1} = b^2$ then determine the order of b.