Math 343 - Intro. to Modern Algebra Homework 4 Due: Feb. 22nd, 2019

Note: A random subset of these problems will be graded for credit.

(1) Let R be a ring in which $x^2 = x$ for every $x \in R$. Show that R is commutative.

(2) Let R be a ring and $a \in R$ be a fixed element. Let $I_a = \{x \in R | ax = 0\}$. Show that I_a is a subring of R.

(3) Let $p \in \mathbb{N}$ be an odd prime. Show that 1 and $p - 1$ are the only elements of the field $\mathbb{Z}_p$ that are their own multiplicative inverses.

Hint: Consider the equation $x^2 - 1 = 0$.

(4) Let R be a ring. Prove each of the following:

(a) If $a^2 = 0$ for some $a \in R \setminus \{0\}$, then $ax + xa$ commutes with a for all $x \in R$.

(b) If R is an integral domain then for all $a, b, c \in R$, $ab = ac$ implies $a = 0$ or $b = c$.

(c) If R is a finite integral domain then R is a field.

Hint: Consider a non-zero element $a \in R$ and the function $m_a : R \to R$ given by $m_a(b) = ab$.

(5) Find the following remainders:

(a) 7^{93} when divided by 11.

(b) 5^{101} when divided by 16.

(6) Show that $\{f : [0, 1] \to \mathbb{R} | f \text{ is differentiable on } (0, 1)\}$ under point-wise addition and multiplication is a commutative ring with unity, but not an integral domain. You can take it for granted that the sum/product of functions preserves differentiability.

(7) Let R be a ring. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$.

(a) Prove that if an element $a \in R$ is nilpotent then either $a = 0$ or a is a zero-divisor.

(b) Suppose R is commutative. Show that if $a \in R$ is nilpotent then so is ra for all $r \in R$.

(c) Suppose R is commutative. Show that if $a, b \in R$ are nilpotent then so is $a + b$.

(d) Suppose $1 \in R$ (that is, R has unity). Prove that if $a \in R$ is nilpotent then $1 + a$ is a unit (has a multiplicative inverse).

Hint: Think about $\dfrac{1}{1+x}$ represented as a power series.

(8) Let F be any (non-trivial) field. Let $R = M_2(F)$ be the ring of 2×2 matrices over F. Show that there are at least 6 units in R.

Hint: F contains at least two elements, 0 and 1.

(9) Let R be a commutative ring of prime characteristic p. Show that the function $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a ring homomorphism.