Math 343 - Intro. Modern Algebra **Homework 5** **Due: March 8th, 2019**

Note: A random subset of these problems will be graded for credit.

(1) Consider $f(x) = 4x^3 + x^2 + 2$ and $d(x) = 4x^2 + 1$ as polynomials in $\mathbb{Z}_5[x]$. Find polynomials $q(x), r(x) \in \mathbb{Z}_5[x]$ such that $f(x) = q(x)d(x) + r(x)$ and the degree of $r(x)$ is less than 2.

(2) Let R be a ring. Let I, J be ideals of R and A be a subring of R.

(a) Show that the $I \cap J$ is an ideal of R.

(b) Show that $I + J = \{a + b | a \in I, b \in J\}$ is an ideal of R.

(c) Show that $I \cap A$ is an ideal of A.

(3) Let R be the ring of 2×2 matrices over $\mathbb{R}$. Suppose I is an ideal of R. Show that $I = (0)$ or $I = R$.

Hint: Recall from linear algebra that the reduced row echelon form of a matrix A is EA for some matrix E, which is a product of elementary matrices (corr. to the row operations).

(4) Let R be a commutative ring and I be an ideal of R. The set

$$\sqrt{I} = \{x \in R | x^n \in I \ \exists n \in \mathbb{Z}^+\},$$

is called the <u>radical</u> of I.

(a) Show that $I \subseteq \sqrt{I}$.

(b) Show that $\sqrt{I}$ is an ideal of R.

(5) Let F be a field and $p(x) \in F[x]$ be of degree 3. Prove that $p(x)$ is irreducible over F if and only if $\forall r \in F$, $p(r) \neq 0$.

(6) Find all irreducible polynomials of...

(a) ... degree 3 in $\mathbb{Z}_2[x]$.

(b) ... degree 2 in $\mathbb{Z}_3[x]$.

(c) ... the form $x^2 + ax + b$ in $\mathbb{Z}_5[x]$.

(7) Let R be a commutative ring. Recall that $a \in R$ is nilpotent if $a^n = 0$ for some positive integer in n.

(a) Show that the set of all nilpotent elements in R forms an ideal (called the nilradical of R).

(b) Show that if N is the nilradical of R then the nilradical of R/N is the trivial ideal $\{0 + N\}$.

(8) Let F be a finite field with more than 2 elements. Prove that the sum of all the elements of F is 0.

Hint: For any non-zero element $a \in F$, the function $f_a : F \to F$ given by $f_a(x) = ax$ is a bijection.

(9) Let F be a field. Show that in the ring $R = M_2(F)$, the set of matrices $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in F \right\}$ is a right ideal (additive subgroup of R such that $\forall s \in S, \forall r \in R,\ sr \in S$) but not a left ideal (additive subgroup of R such that $\forall s \in S, \forall r \in R,\ rs \in S$).