Note: A random subset of these problems will be graded for credit.

(1) Give an example of a group $G$ for which $H = \{a \in G|a^2 = e\}$ is NOT a subgroup.

(2) Prove that if $G$ is a non-trivial group and it has no non-trivial proper subgroups then it is cyclic and of prime order.

(3) If $H, K$ are subgroups of $G$ and $hKh^{-1} \subseteq K$ for all $h \in H$ then $HK \leq G$ (that is, $HK$ is a subgroup).

(4) Let $G$ be a group and $H \leq G$. For $x \in G$ we have $Hx = \{hx|h \in H\}$ (called a right-coset of $H$ in $G$). Show that for all $a, b \in G$ either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

(5) Let $n \geq 2$ be an integer. For each set of matrices, determine if it is a subgroup of the general linear group, $GL_n = \{A \in M_n|\text{det}(A) \neq 0\}$ (invertible $n \times n$ matrices). Justify your answers.

\begin{itemize}
  \item[(a)] Invertible $n \times n$ matrices with no 0s on the main diagonal.
  \item[(b)] All $n \times n$ matrices whose determinant is positive.
\end{itemize}

(6) Let $G$ be a group and $g \in G$. Show that $\phi_g : G \rightarrow G$ given by $\phi_g(a) = gag^{-1}$ is an isomorphism.

(7) Let $G$ be a group. Prove that the map $\text{sq} : G \rightarrow G$ given by $\text{sq}(g) = g^2$ is a homomorphism if and only if $G$ is abelian.