(1) Give an example of a group \( G \) for which \( H = \{ a \in G | a^2 = e \} \) is NOT a subgroup.

Consider \( S_3 = \{ \text{Id}, (12), (13), (23), (123), (132) \} \). Then \( H = \{ a \in S_3 | a^2 = e \} \) would be the set \( \{ \text{Id}, (12), (13), (23) \} \), which is not a group because the operation isn’t closed on this set. Just consider that \( (12)(13) = (123) \) which is not in the set (here we are using the convention that the elements act on the right).

(2) Prove that if \( G \) is a non-trivial group and it has no non-trivial proper subgroups then it is cyclic and of prime order.

Proof: Let \( G \) be a non-trivial group with no non-trivial proper subgroups. Let \( g \in G \) be a non-identity element. Consider \( \langle g \rangle \), the cyclic subgroup generated by \( g \).

Since \( G \) has no non-trivial proper subgroups, \( G = \langle g \rangle \). So \( G \) is cyclic. Let \( \ell > 1 \) be the order of \( G \). Suppose it is composite, that is, there exists integers \( m > 1 \) and \( n > 1 \) such that \( \ell = mn \). Then \( g^m \) would generate a cyclic subgroup of order \( n \), which would be a non-trivial proper subgroup. That contradiction implies \( \ell \) is prime. \( \square \)

(3) If \( H, K \) are subgroups of \( G \) and \( hKh^{-1} \subseteq K \) for all \( h \in H \) then \( HK \leq G \) (that is, \( HK \) is a subgroup).

Proof: Let \( G \) be a group and \( H, K \) be subgroups of \( G \) such that \( hKh^{-1} \subseteq K \) for all \( h \in H \). Obviously \( HK \neq \emptyset \). Let \( x, y \in HK \). Then \( x = h_1k_1 \) and \( y = h_2k_2 \) for some \( h_1, h_2 \in H \) and \( k_1, k_2 \in K \). Let’s attempt (1-step subgroup test) to show that \( xy^{-1} \in HK \). Well,

\[
xy^{-1} = h_1k_1k_2^{-1}h_2^{-1} = (h_1h_2^{-1})(h_2k_1k_2^{-1}h_2^{-1}) \in HK,
\]

since \( h_1h_2^{-1} \in H \) and \( h_2k_1k_2^{-1}h_2^{-1} \in K \) using that \( hKh^{-1} \subseteq K \) for all \( h \in H \). \( \square \)

(4) Let \( G \) be a group and \( H \leq G \). For \( x \in G \) we have \( Hx = \{ hx | h \in H \} \) (called a right-coset of \( H \) in \( G \)). Show that for all \( a, b \in G \) either \( Ha = Hb \) or \( Ha \cap Hb = \emptyset \).

Proof: Let \( a, b \in G \). Suppose \( Ha \cap Hb \neq \emptyset \). So there exists \( h_1, h_2 \in H \) such that \( h_1a = h_2b \). But then \( a = h_1^{-1}h_2b \). Let \( x \in Ha \). Then \( x = ha \) for some \( h \in H \). By substitution, \( x = hh_1^{-1}h_2b \in Hb \). So \( Ha \subseteq Hb \). A similar argument, using \( b = h_2^{-1}h_1a \) yields \( Hb \subseteq Ha \). Hence \( Ha = Hb \). So either \( Ha = Hb \) or \( Ha \cap Hb = \emptyset \). \( \square \)
(5) Let $n \geq 2$ be an integer. For each set of matrices, determine if it is a subgroup of the general linear group, $GL_n = \{ A \in M_n | \det(A) \neq 0 \}$ (invertible $n \times n$ matrices). Justify your answers.

(a) Diagonal $n \times n$ matrices with no $0$s on the main diagonal.

$I_n$ is a $n \times n$ diagonal matrix with no $0$s on the main diagonal. Let $D_1, D_2$ be $n \times n$ diagonal matrices with $0$s on the main diagonal. Then they are elements of $GL_n$ as their determinants (product of the diagonal entries) are non-zero. We know (from linear algebra) that the product of two diagonal matrices is once again a diagonal matrix (where each main diagonal entry is the product of the two corresponding diagonal entries). Let the main diagonal of $D_1$ be $\lambda_1, \lambda_2, ..., \lambda_n$ and the main diagonal of $D_2$ be $\lambda'_1, \lambda'_2, ..., \lambda'_n$. Since $\lambda'_i \neq 0$ for $i = 1, 2, ..., n$ we have that $D_2^{-1}$ is the diagonal matrix with main diagonal $1/\lambda'_1, 1/\lambda'_2, ..., 1/\lambda'_n$, and hence has no $0$s on the main diagonal. Then $D_1D_2^{-1}$ is a diagonal matrix with main diagonal $\lambda_1/\lambda'_1, \lambda_2/\lambda'_2, ..., \lambda_n/\lambda'_n$ and has no $0$s on the main diagonal since $\lambda_i \neq 0$ for $i = 1, 2, ..., n$. Therefore diagonal $n \times n$ matrices with no $0$s on the main diagonal form a subgroup of $GL_n$.

(b) All $n \times n$ matrices whose determinant is positive.

$I_n$ is an $n \times n$ matrix with a positive determinant as $\det(I_n) = 1$. Let $A, B$ be $n \times n$ matrices with positive determinants. Clearly they are elements of $GL_n$. We know (from linear algebra) that $\det(B^{-1}) = 1/\det(B)$. Since $\det(B) > 0$ it follows that $\det(B^{-1}) > 0$. Therefore, $\det(AB^{-1}) = (\det(A)) (\det(B^{-1}))$ is positive since $\det(A) > 0$ and $\det(B^{-1}) > 0$. Thus $n \times n$ matrix with a positive determinant form a subgroup of $GL_n$.

(6) Let $G$ be a group and $g \in G$. Show that $\phi_g : G \to G$ given by $\phi_g(a) = gag^{-1}$ is an isomorphism.

**Proof:** Let $\phi_g : G \to G$ be given by $\phi_g(a) = gag^{-1}$. Let $a, b \in G$. Suppose $\phi_g(a) = \phi_g(b)$. Then $gag^{-1} = bgb^{-1}$. By left- and right-cancelation it follows that $a = b$. So $\phi_g$ is $1 - 1$. Let $c \in G$. Then $g^{-1}cg \in G$ and $\phi_g(g^{-1}cg) = g(g^{-1}cg)g^{-1}$ which simplifies to $c$. Hence $\phi_g$ is onto, and thus a bijection. Furthermore, $\phi_g(ab) = g(ab)g^{-1} = ga(g^{-1}g)bg^{-1} = (gag^{-1})(gbg^{-1}) = \phi_g(a)\phi_g(b)$, showing that $\phi_g$ is a homomorphism. Therefore $\phi_g$ is an isomorphism. □

(7) Let $G$ be a group. Prove that the map $sq : G \to G$ given by $sq(g) = g^2$ is a homomorphism if and only if $G$ is abelian.

**Proof:** Let $sq : G \to G$ given by $sq(g) = g^2$. Suppose $sq$ is a homomorphism. Let $g, h \in G$. Then $sq(gh) = (gh)^2$ and $sq(gh) = sq(g)sq(h) = g^2h^2$. So $ghgh = gghh$, which implies $h = gh$ by left multiplication by $g^2$ and right multiplication by $h^{-1}$. So $G$ is abelian. Now suppose that $G$ is abelian. Then $sq(gh) = (gh)^2 = ghgh = gghh = g^2h^2 = sq(g)sq(h)$. So $sq$ is a homomorphism. □