

# Adaptive Learning in Regime-Switching Models\*

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## Abstract

This paper studies adaptive learning in economic environments subject to recurring structural change. Stochastically evolving institutional and policy-making features can be described by regime-switching rational expectations models whose parameters evolve according to a finite state Markov process. We demonstrate that in non-linear models of this form, two natural schemes emerge for learning the conditional means of endogenous variables: under *mean value learning*, the equilibrium's lag structure is assumed exogenous and therefore known to agents; whereas, under *vector autoregression learning* (VAR learning), the equilibrium lag structure depends endogenously on agents' beliefs and must be learned. We show that an intuitive condition, analogous to the 'Long-run Taylor Principle' of Davig and Leeper (2007), ensures convergence to a regime-switching rational expectations equilibrium. However, the stability of sunspot equilibria, when they exist, depends on whether agents adopt mean value or VAR learning. Coordinating on sunspot equilibria via a VAR learning rule is not possible. These results show that, when assessing the plausibility of rational expectations equilibria in non-linear models, out of equilibrium behavior is important.

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# 1 Introduction

A given forward-looking macroeconomic model may admit different classes of rational expectations equilibria. Solutions can differ in terms of the set of state variables that agents use when forming expectations. For example, standard linear stochastic rational expectations models have solutions that depend only on the minimal set of state variables, but may also have solutions that depend on extrinsic random variables (i.e. sunspots). The existence of equilibria are well understood in linear models with constant parameters; however, in a growing area of research that focuses on models with changing parameters, these issues are re-emerging.<sup>1</sup> In regime-switching models, which constitute the focus of this paper, parameters evolve according to a finite state Markov process. The non-linear structure of regime-switching rational expectations models prevents a complete characterization of the full class of solutions. Two recent papers by Davig and Leeper (2007) and Farmer, Waggoner and Zha (2007) each focus on a particular class of equilibria in the context of a standard New Keynesian model with a monetary policy rule whose coefficients are subject to occasional regime change.

As a means of addressing issues of multiple equilibria in non-linear models, this paper studies stability under adaptive learning of rational expectations equilibria in regime-switching models. Our approach begins by generalizing and extending the previous work on existence of equilibria in multivariate regime-switching models by defining two classes of solutions. The distinguishing feature between the two classes is whether the resulting equilibrium's conditional distribution exhibits explicit dependence on both current and lagged regimes. To fix terminology, we define the class that restricts lagged regimes from entering the state vector as *Regime-Dependent Equilibria* (RDE) and the other class, where lagged regimes enter the state vector, as *History-Dependent Equilibria* (HDE). HDE generally admit sunspot shocks, regardless of the parameterization of the model. Davig and Leeper (2007) introduce a condition known as the Long Run Taylor Principle (LRTP) that ensures a unique RDE, whereas Farmer, Waggoner and Zha (2007) expand on this work by constructing an HDE that admits sunspot shocks even when the LRTP holds. Adapting the LRTP condition to a general multivariate setting, we define the Conditionally Linear Determinacy Criterion (CLDC) as the condition guaranteeing the existence of a unique RDE. This paper, importantly, establishes a connection between the learnability of equilibria and the CLDC.

The results above distinguish regime-switching models from their constant parameter counterparts. In particular, conditions that establish uniqueness of equilibria

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<sup>1</sup>Some examples of work in this area include Leeper and Zha (2003), Andolfatto and Gomme (2003), Davig (2004), Zampolli (2006), Chung, Davig and Leeper (2007), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2006, 2007), and Svensson and Williams (2007). Brainard (1967) is an early example of work on parameter instability.

within a certain natural class may fail to preclude the existence of other types of equilibria. Following Lucas (1986), we maintain that stability under adaptive learning is a useful metric for identifying empirically relevant equilibria.<sup>2</sup> An equilibrium is plausible or reasonable, if whenever rational expectations are replaced with a standard adaptive learning rule, agents' beliefs converge to the rational expectations equilibrium values. Based on this assumption, we assess whether regime-switching equilibria are learnable.

Our viewpoint is informed by a large and growing literature that replaces rational expectations with learning rules where agents are modeled as professional econometricians, that is, they hold forecasting models that share a reduced-form with a rational expectations equilibrium, and adjust the parameters of their model in light of new data. The advantage to this approach is that it places economist and agent on equal footing and avoids the cognitive dissonance inherent in rational expectations models. This approach is particularly compelling in regime-switching models because of the co-existence of equilibria in the regime and history dependent classes.

In this paper, we endow agents with a learning algorithm, similar in reduced-form to the equilibria identified by Davig and Leeper (RDE) and Farmer, Waggoner and Zha (HDE), and we study the stability of the associated equilibria. It is well known that in models with multiple equilibria, different learning rules (each *a priori* plausible) may lead to distinct stability outcomes.<sup>3</sup> Sticking with the statistical learning approach outlined above, two natural learning models emerge for history dependent equilibria. These learning processes differ based on basic informational assumptions. In the first formulation – “mean value learning” – agents know the lag structure of the economy's endogenous variables when estimating the conditional mean. The second natural learning process – “VAR learning” – agents try also to uncover the lag structure from the data by employing a (first order) vector autoregression (VAR(1)) model. Within a rational expectations equilibrium the two formulations are equivalent, but out of equilibrium they lead to different conclusions about the learnability of HDE.

Our primary result is that when the CLDC is satisfied (i.e. the Long-run Taylor Principle in a New Keynesian model) there exists a unique Regime-dependent Equilibrium and that equilibrium is stable under learning. Moreover, this condition also governs the stability of HDE under mean value learning, that is, when agents know the lag structure of the model, but estimate in real-time the mean. On the other hand, in a univariate model and in a New Keynesian model, the HDE are not attainable under VAR learning.

The results of this paper provide a clear illustration of the usefulness of expectational stability as an equilibrium selection criterion. As the results of Davig and

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<sup>2</sup>See, in addition, Evans (1986), Bray and Savin (1986), Marcet and Sargent (1989).

<sup>3</sup>See Lucas (1986), Woodford (1990).

Leeper (2007) and Farmer, Waggoner, and Zha (2007) demonstrate, local determinacy is not particularly useful in these settings for selecting equilibria and designing policy. This paper shows that the CLDC ensures that a rational expectations equilibrium can be attained as the limiting outcome of a reasonable learning process. Moreover, if agents behave as econometricians and try to uncover the lag structure of the endogenous state variables, then the CLDC guarantees the existence of a unique stable equilibrium.

The paper is organized as follows : Section 2 introduces the framework of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2007); Section 3 generalizes the model and defines the classes of equilibria; Section 4 provides the main stability analysis; Section 5 presents results for a univariate and a New Keynesian model; and, Section 6 concludes.

## 2 A New Keynesian Model with Recurring Policy Change

There is extensive empirical evidence of regime change in monetary policymaking. For example, there is a breakpoint in the parameters of a Taylor-type nominal interest rate rule in Clarida, Gali, and Gertler (1999), or shifting policymaker preferences in Bernanke (2004) and Dennis (2006). These findings motivate models that build regime-switching directly into rational expectations frameworks since whenever the structural nature of monetary policymaking has changed in the past, then it is reasonable that agents might anticipate future policy changes.

As an example, Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2007) construct rational expectations solutions to the standard New Keynesian model closed with a nominal interest rate rule whose coefficients are subject to occasional regime change. The New Keynesian model is given by (linearized) reduced-form equations for inflation,  $\pi$ , and the output gap,  $x$ , such as

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + g_t \quad (1)$$

$$x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + u_t \quad (2)$$

which is closed with a nominal interest rate rule with time-varying parameters

$$i_t = \alpha_t \pi_t + \gamma_t x_t \quad (3)$$

To capture recurrent regime change, Davig and Leeper (2007), assume that the parameters  $\alpha_t, \gamma_t$  in (3) follow a two state Markov chain:

$$\alpha_t = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2 \end{cases}$$

and

$$\gamma_t = \begin{cases} \gamma_1 & \text{for } s_t = 1 \\ \gamma_2 & \text{for } s_t = 2 \end{cases}$$

The random variable  $s_t$  follows a finite-state Markov chain with transition probabilities  $p_{ij} \equiv \Pr [s_t = j | s_{t-1} = i]$  for  $i, j = 1, 2$ .

The definition of a rational expectations equilibrium in regime-switching models is analogous to the concept in constant parameter models. In particular, a *rational expectations equilibrium* is any uniformly bounded solution to (1)-(3). Because the Markov chain enters the model multiplicatively, the model is inherently non-linear, preventing a characterization of all solutions to the model. Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2007) propose two classes of solutions. Farmer, Waggoner, and Zha (2007) show that it is possible for there to exist a continuum of solutions in one class even under conditions that guarantee uniqueness in the other class. This raises the question of which equilibria are plausible. This provides one motivation for studying adaptive learning as an equilibrium selection device.

### 3 Equilibria In Regime Switching Models

We focus on models whose reduced form consists of a system of non-linear expectational difference equations such as

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t, \tag{4}$$

$$r_t = \rho r_{t-1} + \varepsilon_t, \tag{5}$$

where  $y_t$  is an  $(n \times 1)$  vector of random variables,  $\beta_t$  and  $\gamma_t$  are conformable matrices that follow an  $m$  state Markov process with  $(\beta_t = \beta_i, \gamma_t = \gamma_i) \Leftrightarrow s_t = i, i = 1, 2, \dots, m$ , and  $r_t$  is a  $(k \times 1)$  exogenous stationary VAR(1) process independent of  $s_j$  for all  $j$ . The stochastic matrix  $P$  governs the evolution of the state,  $s_t$ , and contains elements

$$p_{ij} \equiv \Pr [s_t = j | s_{t-1} = i],$$

for  $i, j \in \{1, 2, \dots, m\}$ .  $P$  is taken to be recurrent and aperiodic, so that it has a unique stationary distribution  $\Pi$ . For simplicity,  $\beta_i$  is taken to be invertible for all  $i$ . Davig and Leeper (2007) consider a version of this model in the context of a univariate monetary model and a bivariate New Keynesian model. Most macroeconomic models feature expectational structures similar to (4) – albeit with constant parameters – making (4) a natural laboratory to study the existence and stability of rational expectations equilibria in regime-switching models. In Section 5, we present a univariate example and return to the New Keynesian example.

A rational expectations equilibrium of the model is a solution to (4) that also satisfies a boundary condition. Often the definition of the boundary condition is somewhat vague, given as “non-explosiveness” and justified by appealing to a transversality condition, even though the usual transversality condition implies that solutions not explode “too quickly.”<sup>4</sup>

We focus on processes satisfying the following property:

**Definition.** A stochastic process  $y_t$ , with initial condition  $y_0$  is *uniformly bounded* (almost everywhere) or UB if  $\exists M(y_0)$  so that  $\sup_t \|y_t\|_\infty < M(y_0)$ , where  $\|\cdot\|_\infty$  is the  $L^\infty$  or “essential supremum” norm.

With this definition available, we may define a rational expectations equilibrium:

**Definition.** A *Rational Expectations Equilibrium* is any UB stochastic process satisfying (4).

While uniformly bounded (UB) may appear to be an *a priori* strong notion of boundedness, it is common in the linear rational expectations literature. In linear models with constant parameters, uniform boundedness is consistent with the usual notion of model determinacy, such as in Blanchard and Kahn (1980). Also, UB “bounds the paths” of all endogenous variables and is often desirable when using a first-order approximation to a nonlinear model around a fixed point, such as a steady state.

An important difference that arises in regime-switching rational expectations models, versus constant-parameter models, is that agents incorporate the probability of a regime change into their expectations. The resulting non-linear structure prevents characterizing the full class of rational expectations equilibria. However, several classes naturally emerge, which we define as *Regime-Dependent Equilibria* (RDE), *Stacked System Equilibria*, and *History-Dependent Equilibria* (HDE).

### 3.1 Regime-Dependent Equilibria

The first class focuses on state-contingent solutions that allow the current realization of the regime,  $s_t$ , to enter the state vector, but are otherwise independent of its history. The state vector also includes current realizations of the exogenous shocks, as well as (possibly) sunspot variables. Formally, the definition for an RDE is as follows:

**Definition.** Let  $s_t$  be the Markov process governed by  $P$  and taking values in  $\{1, 2, \dots, m\}$ . Let  $y_t$  be a solution to (4). Then  $y_t$  is a *Regime Dependent Equilib-*

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<sup>4</sup>As an alternative, sometimes the boundary condition requires the paths of variables in a rational expectations equilibrium remain conditionally uniformly bounded, such as in Evans and McGough (2005).

rium (RDE) if it is uniformly bounded and there exist uniformly bounded stochastic processes  $y_{1t}, y_{2t}, \dots, y_{mt}$ , with  $y_{it}$  independent of  $s_{t+j}$  for all integers  $j$ , such that  $y_t = y_{it} \Leftrightarrow s_t = i$ .

In an RDE, depending on the realization of  $s_t$ ,  $y_t$  takes on values from one of  $m$  stochastic processes, with each process being independent of the Markov state. Conditioning (4) on each regime leads to the following system

$$\begin{aligned} y_{1t} &= \beta_1 p_{11} E_t y_{1t+1} + \beta_1 p_{12} E_t y_{2t+1} + \dots + \beta_1 p_{1m} E_t y_{mt+1} + \gamma_1 r_t, \\ y_{2t} &= \beta_2 p_{21} E_t y_{1t+1} + \beta_2 p_{22} E_t y_{2t+1} + \dots + \beta_2 p_{2m} E_t y_{mt+1} + \gamma_2 r_t, \\ &\vdots \\ y_{mt} &= \beta_m p_{m1} E_t y_{1t+1} + \beta_m p_{m2} E_t y_{2t+1} + \dots + \beta_m p_{mm} E_t y_{mt+1} + \gamma_m r_t, \end{aligned}$$

which governs dynamics for  $y_{it}$  for  $i = 1, 2, \dots, m$ . We note that this is a linear system.

### 3.2 Stacked System Equilibria

The linear system above can be recast in the form of a ‘stacked system’, which has a more compact representation. Stacked System Equilibria are rational expectations solutions to the conditionally linear system, without the restriction of independence from  $s_{t+n}$  for all  $n$  that is explicitly imposed on RDE.

**Definition.** The *stacked system* associated with the switching model (4) is the system of multivariate linear expectational difference equations

$$\hat{y}_t = (\oplus_{j=1}^m \beta_j)(P \otimes I_n) E_t \hat{y}_{t+1} + \gamma r_t \quad (6)$$

where  $\hat{y}_t = [y'_{1t}, y'_{2t}, \dots, y'_{mt}]'$  and  $\gamma' = (\gamma'_1, \dots, \gamma'_m)'$ .<sup>5</sup>

**Definition.** Let  $s_t$  be the Markov process governed by  $P$  and taking values in  $\{1, 2, \dots, m\}$ . Let  $y_t$  be a solution to (4). Let  $\hat{y}_t$  be a uniformly bounded solution to the Stacked System (6). Then  $y_t$  is a *Stacked System Equilibrium* (SSE) of the original model (4) if  $y_t = \hat{y}_{it} \Leftrightarrow s_t = i$ .

The stacked system (6) is a multivariate linear rational expectations model. The number and nature of solutions to (6) is well-known. We are particularly interested in conditions under which there exists a unique UB solution to the stacked system, as this condition will also govern the expectational stability of solutions to the economic system (4). We summarize this in the following remark.

**Remark.** A necessary and sufficient condition for the existence of a unique uniformly bounded solution to (6) is that the eigenvalues of  $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$  lie inside the unit

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<sup>5</sup>Throughout,  $\oplus$  denotes the direct sum operator.

circle. In this case, we say that the *Conditionally Linear Determinacy Condition* (CLDC) is satisfied.

The following proposition summarizes the relationship between solutions to the stacked system (6), RDE, and UB solutions to (4).

**Proposition 1** *Let  $U$  be the collection of all UB solutions to (4). Let  $R, S$  denote the collection of all Regime Dependent Equilibria and all UB solutions to the Stacked System (6), respectively. The following relations among the sets of equilibria hold.*

1.  $R \subset S \subset U$ .
2. *If the CLDC holds, then  $R = S$  and  $R$  has one element.*
3. *If the CLDC does not hold, then  $R$  has a continuum of elements and  $R \subsetneq S$*

All proofs are contained in the Appendix.

In the context of monetary policy analysis, Davig and Leeper (2007) refer to the CLDC as the *Long Run Taylor Principle* (LRTP). In this respect, Davig and Leeper (2007) completely characterize the uniqueness conditions for RDE in standard monetary models. Davig and Leeper, however, restrict attention to models with positive feedback from monetary policy so that the eigenvalues of  $\oplus\beta_j$  are positive. The results in Proposition 1 establish that the necessary and sufficient condition for existence of a unique RDE is the CLDC. Importantly, however, when the CLDC is not satisfied there may exist other equilibria that are not independent of past realizations of the Markov state. Below we call such equilibria *History Dependent Equilibria*. Farmer, Waggoner, and Zha (2007) show that the CLDC does not imply uniqueness in the economic model (4). In the next Section, we propose expectational stability as a device for selecting equilibria when both RDE and HDE exist.

Subsequent sections show a close connection between the conditions for unique RDE and E-stable rational expectations equilibria, and so the CLDC takes on added importance below.

### 3.3 History Dependent Equilibria

Proposition 1 shows that the collection of uniformly bounded solutions to (4) supercedes the set of regime dependent equilibria. This section characterizes another class of equilibria to (4). This definition is related to the work of Farmer, Waggoner, and Zha (2007) who illustrate that in a New Keynesian model, it is possible for there to exist sunspot equilibria even when the RDE is unique. In this case, conditions ensuring

uniqueness within the class of RDE, such as the CLDC, does not ensure equilibrium determinacy in the regime-switching rational expectations framework. The central element in Farmer, Waggoner, and Zha (2006, 2007) (FWZ), is they allow lagged states to enter the state vector. That is, FWZ have agents conditioning their expectations on an expanded state vector that includes  $s_{t-1}$ . For this reason, we call the class of solutions *History Dependent Equilibria* (HDE). By assuming agents condition on current and past realizations of the state variable  $s_t$ , this class of equilibria includes solutions that depend on arbitrary sunspot variables.

**Definition.** Let  $s_t$  be the Markov process governed by  $P$ , taking values in  $\{1, 2, \dots, m\}$ . Let  $y_t$  be a solution to (4). Then  $y_t$  is a *History Dependent Equilibrium* (HDE) if it is uniformly bounded and its distribution conditional on  $s_t$  differs from its distribution conditional on  $s_t$  and  $s_{t-1}$ ; that is,  $y_t|s_t \not\sim y_t|(s_t, s_{t-1})$ .

**Remark.** The definition of an HDE restricts solutions to the class of uniformly bounded stochastic processes whose conditional density exhibits dependence on  $s_t$  and  $s_{t-1}$ . Notice that if  $y_t$  is an RDE then  $y_t|s_t \sim y_t|(s_t, s_{t-1})$ . However, by Proposition 1 when the CLDC is not satisfied, then there may exist solutions to the stacked system that are not RDE. In particular, when the matrix  $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$  has  $n_s$  eigenvalues inside the unit circle then for each  $n_s$ -dimensional martingale difference sequence  $\xi_t$  there is a martingale difference sequence  $\tilde{\xi}_t$  and an SSE  $\hat{y}_t$  with a representation given by

$$\hat{y}_t = b\hat{y}_{t-1} + cr_{t-1} + d(s_{t-1}, s_t)\tilde{\xi}_t,$$

where  $d$  is any function of  $s_{t-1}$  and  $s_t$ .

HDE solutions to the stacked system have constant parameters except for the coefficient on the sunspot shock  $\tilde{\xi}_t$ . It is natural to wonder whether there exist HDE with time-varying coefficients on the lagged endogenous variable. To consider this, we assume  $\gamma_t = 0$  for all  $t$ , as the presence of exogenous shocks does not alter the results and distracts from the presentation. Note that if  $\xi_t$  is any martingale difference sequence, then  $y_t = \beta_{t-1}^{-1}y_{t-1} + \xi_t$  is a solution to (4). Farmer, Waggoner, and Zha (2007) show that there exist multiple uniformly bounded HDE that have the following representation

$$y_t = \left( \frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) y_{t-1} + v_{s_t} \xi_t, \quad (7)$$

provided there exists  $c_1, \dots, c_m$  and  $v = (v'_1, \dots, v'_m)' \neq 0$  so that  $|c_j| \leq 1$  and  $c$  and  $v$  solve

$$\left[ \left( \oplus_{j=1}^m \beta_j \right)^{-1} - \left( \left( \oplus_{j=1}^m c_j \right) P \right) \otimes I_n \right] v = 0. \quad (8)$$

Here  $\xi_t$  is independent of  $s_{t+n}$  for all  $n$ . The condition (8) is essentially derived from the method of undetermined coefficients. When (8) is satisfied, solutions to

the representation (7) are solutions to (4).<sup>6</sup> The construction of the autoregressive parameter in the representation (7) is chosen so that, regardless of the history of realizations of  $s_t$ , these parameters are bounded in matrix norm and, hence, the solutions are uniformly bounded.

Farmer, Waggoner, and Zha (2007) write (7) in alternative form

$$y_t = \eta_t \tag{9}$$

where

$$\eta_t = \left( \frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) \eta_{t-1} + v_{s_t} \xi_t.$$

The stochastic properties of (9) are equivalent to (7); however, as we will see below, during out of equilibrium learning dynamics, these two representations imply different informational assumptions and distinct stability results. These observations lead to two natural learning rules: a “mean value learning” formulation where agents use a forecasting model consistent with (9) by *conditioning* on  $\eta_t$  and try to learn the endogenous variable’s state-contingent constant term (which, in this case, is zero); and a “VAR learning” formulation where agents estimate a forecasting model consistent with (7) by *conditioning* on a state-contingent constant and on lagged  $y$  – in this case, agents must also learn the endogenous variable’s lagged coefficients. Importantly, under mean value learning, the lag structure is exogenous while it is determined endogenously under VAR learning. This provides a crucial distinction for the stability results presented below.

By defining HDE as rational expectations equilibria that exhibit conditional dependence on both  $s_t$  and  $s_{t-1}$ , it is possible to identify a more general class of equilibria than those represented by (9). Assume HDE take the form<sup>7</sup>

$$y_t = B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t. \tag{10}$$

where the coefficients must satisfy

$$\left( I_n - \beta_j \left( \sum_{k=1}^m p_{jk} B_{jk} \right) \right) B(i, j) = 0 \tag{11}$$

$$\left( I_n - \beta_j \left( \sum_{k=1}^m p_{jk} B_{jk} \right) \right) C(i, j) = 0 \tag{12}$$

Notice that provided non-zero  $B(i, j)$  satisfy (11), the  $C(i, j)$  are arbitrary. It is straightforward to verify that (7) is a solution to (10).

<sup>6</sup>If one were to literally use the method of undetermined coefficients, the  $v$  in (8) would be  $y_t$ . However, if  $v$  is taken to be a vector of initial conditions chosen to lie on the stable manifold, and if (8) is satisfied at  $t = 1$ , then it will be satisfied for all  $t$ .

<sup>7</sup>Adopting the earlier notation, since  $\gamma_t = 0$  it follows that  $\tilde{\xi}_t = \xi_t$ .

## 4 Equilibrium Selection in Regime-Switching Models: Expectational Stability

Although rational expectations solutions to regime-switching models are of interest to applied economists and policymakers, the technical details leave important practical issues unsettled. First, as illustrated in the above section, the concepts of determinacy and uniqueness of rational expectations equilibria in non-linear models are not readily available. The RE hypothesis, in reduced-form models, is silent about which class of equilibria is most reasonable. Second, by imposing rational expectations, the modeler makes strong assumptions that require private-sector agents know the true distribution generating the data, even though the model is self-referential. Applied economists and professional forecasters typically formulate reduced-form models, inspired by rational expectations equilibria, that they estimate based on available data and update as new data becomes available. It is reasonable to expect that private-sector agents behave similarly.

A somewhat recent literature on adaptive learning in macroeconomics studies the plausibility of rational expectations equilibria by insisting on logical consistency between professional forecasters (or econometricians) and private-sector agents. Rather than rational expectations, this literature assumes that agents behave as econometricians who formulate forecasting models and update the parameters of their model in real-time. Because the data generating process depends on these recursively updated forecasting models, the convergence to, and stability of, rational expectations equilibria is a non-trivial problem. Woodford (1990), Marcet and Sargent (1989), Evans and Honkapohja (2001), Bullard and Mitra (2002) argue that stability under learning is a reasonable equilibrium selection mechanism. We adopt this viewpoint and study the stability under learning of regime-switching rational expectations equilibria. Our primary result is that the condition governing uniqueness in the class of regime dependent equilibria, namely, the CLDC, may also be the condition governing expectational stability. Crucially, though, this result depends on the assumed information structure. Evans and Honkapohja (2008) argue that indeterminacy need not concern policymakers when the fundamentals rational expectations equilibrium is the only equilibrium stable under learning. Adapting this viewpoint to a regime-switching framework suggests the CLDC, or the Long-run Taylor Principle of Davig and Leeper (2007), can form a sensible policy prescription since it may select the desired (unique) RDE.

## 4.1 E-stability in Constant Parameter Models

To fix ideas, we review the expectational stability approach in a constant parameter version of (4),

$$y_t = \beta E_t^* y_{t+1} + \gamma r_t \quad (13)$$

now written with a (possibly) boundedly rational expectations operator  $E^*$ . We first consider the case where the model is determinate and then, below, we examine the indeterminate case.

When the model is determinate, there exists a unique equilibrium that has the form  $y_t = br_t$ . Agents hold a perceived law of motion (i.e. a forecasting model) whose functional form is consistent with the equilibrium representation

$$y_t = A + Br_t. \quad (14)$$

While there is no constant in the equilibrium representation  $y_t = br_t$ , it is standard to allow agents to consider the possibility that there may be a constant term, i.e. to learn the steady-state values of  $y$  as well.

The parameters  $A$  and  $B$  capture agents' perceptions of the relationship between  $y$  and  $r$  and may be estimated using, for example, recursive least squares. Let  $A_t$  and  $B_t$  be the respective estimates using data up to time  $t$ . Agents form forecasts using the perceived law of motion  $E_t^* y_{t+1} = A_{t-1} + B_{t-1} \rho r_t$ . Plugging these forecasts into (13) leads to the actual law of motion

$$y_t = \beta A_{t-1} + (\beta B_{t-1} \rho + \gamma) r_t.$$

Here we assume that agents know the true process governing  $r_t$ . The actual law of motion illustrates the manner in which time  $t$  endogenous variables are determined by perceptions  $(A_{t-1}, B_{t-1})$  and realizations of  $r_t$ . Given new data on  $y_t$  agents then update the forecasting model to obtain  $(A_t, B_t)$ . The unique rational expectations equilibrium  $y_t = br_t$  is stable under learning if  $(A_t, B_t) \rightarrow (0, b)$  almost surely. Stability under learning is non-trivial precisely because of the self-referential nature of rational expectations models. That is, the actual law of motion depends on the perceptions  $A_{t-1}, B_{t-1}$  and convergence is not obvious.

While assessing the asymptotic behavior of the non-linear stochastic process  $(A_t, B_t)$  is quite difficult, it turns out that the technical requirements for convergence often reduce to a fairly simple and intuitive condition known as E-stability, see Evans and Honkapohja (2001). To illustrate, suppose agents hold generic beliefs  $(A, B)$ . The actual law of motion then defines a map  $T : \mathbb{R}^n \oplus \mathbb{R}^{n \times k} \rightarrow \mathbb{R}^n \oplus \mathbb{R}^{n \times k}$  that takes perceived coefficients to actual coefficients

$$T(A, B) = (\beta A, \beta B \rho + \gamma).$$

Notice that the fixed point of the T-map identifies the unique rational expectations equilibrium of the model. The rational expectations equilibrium is said to be E-stable if it is a locally asymptotically stable fixed point of the ordinary differential equation (o.d.e.)

$$\frac{d(A, B)}{d\tau} = T(A, B) - (A, B). \quad (15)$$

The E-stability Principle states that if agents use recursive least squares – or, similar reasonable learning algorithms – then E-stable rational expectations equilibria are locally stable under learning.<sup>8</sup> In this simple example, if  $(0, b)$  is a locally asymptotically stable fixed point of (15) then  $(A_t, B_t) \rightarrow (0, b)$  almost surely.

The economic intuition behind the E-stability principle is that reasonable learning algorithms dictate that agents update their parameter estimates in the direction of forecast errors. This is evident in (15), as  $T(A, B) - (A, B)$  is, in a sense, a forecast error. If the resting point of the o.d.e. is stable then adjusting parameters in the direction of the forecast error will lead the parameters toward the rational expectations equilibrium. Conveniently, conditions for local asymptotic stability are easily computed by examining the eigenvalues of the Jacobian matrix  $DT$ . If all eigenvalues of  $DT$  have real parts less than one then the rational expectations equilibrium is E-stable. For the case at hand, the derivatives are given by  $\beta$  and  $\rho' \otimes \beta$ .<sup>9</sup> Since the model is determinate by assumption, the eigenvalues of  $\beta$  are inside the unit circle and so the rational expectations equilibrium is stable under learning.

If the model is indeterminate then there exists a continuum of equilibria. To analyze stability under learning, we must take a stand on the information available to agents. To fix ideas and avoid unnecessary complications, assume that the model (13) is univariate and non-stochastic ( $r_t = 0$ ). We first assume that agents engage in “mean value learning,” that is, they have knowledge of the endogenous variable’s lag structure, and thus have only the mean to estimate.<sup>10</sup> Specifically, agents condition on the extrinsic process  $\eta_t = \beta^{-1}\eta_{t-1} + \xi_t$ , where  $\xi_t$  is a martingale difference sequence capturing fluctuations in forecast error. The extrinsic noise process  $\eta_t$  captures the serial correlation that arises as a self-fulfilling outcome. By conditioning on  $\eta_t$  the lag structure is imposed exogenously. Agents form expectations using a forecasting model of the form

$$y_t = A + B\eta_t.$$

Computing the T-map provides  $DT_A = \beta$ ,  $DT_B = I$ , so that the sunspot equilibria are E-stable provided the eigenvalues of  $\beta$  are less than  $-1$ .

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<sup>8</sup>The connection between E-stability of a rational expectations equilibrium and its stability under real time learning is quite deep: see Evans and Honkapohja (2001) for details.

<sup>9</sup>Here, and below, we exploit that when the T-map decouples, we can compute derivatives separately. Also, recall that the eigenvalues of the Kronecker product are the products of the eigenvalues.

<sup>10</sup>This method of learning is closely related to common factor representations, see Evans and McGough (2005) for details.

Another natural learning process is “VAR learning” where agents estimate both the mean and the lag structure of the endogenous variables. Specifically, agents condition their forecast on the martingale difference sequence sunspot  $\xi_t$ , as well as a constant and lagged  $y$

$$y_t = A + By_{t-1} + C\xi_t.$$

The primary difference between VAR and mean value learning is that the latter assumes agents identically coordinate on the serially correlated sunspot  $\eta_t$ , while the former postulates that agents try to detect the appropriate lag structure from the data. Under VAR learning, then, the lag structure is determined endogenously. Computing the T-map provides the following derivatives

$$\begin{aligned} DT_A &= \beta(1+b) \\ DT_B &= 2bB \\ DT_C &= \beta b. \end{aligned}$$

Since  $\bar{b} = \beta^{-1}$  it follows that that  $DT_B = 2$ . So if agents employ VAR learning, then the sunspot equilibria are never stable.

This example illustrates that the stability of sunspot equilibria depends on agents’ conditioning set. By incorporating the serial correlation into  $\eta_t$  – which only arises in the model because of self-fulfilling expectations – the agents can coordinate on a sunspot equilibrium. If, however, they are trying to learn the mean and (endogenously determined) lag structure, coordination via learning is not possible.

## 4.2 E-stability in Regime-Switching Models

We now consider the stability properties of regime-switching equilibria.

### 4.2.1 E-stability and the CLDC

This Section demonstrates that the CLDC governs E-stability of RDE. If the CLDC is satisfied, then the unique RDE will have the following minimal state variable representation

$$y_t = B(s_t)r_t. \tag{16}$$

To solve for  $B(s_t)$  for  $s_t \in \{1, 2, \dots, m\}$ , use the stacked system and set  $B = (B(1)', \dots, B(m)')'$ , which yields  $\hat{y}_t = Br_t$ , where

$$\text{vec}(B) = (I_{nm} - \rho' \otimes (\oplus_{j=1}^m \beta_j) (P \otimes I_n))^{-1} \text{vec}(\gamma).$$

It is worth remarking at this point that the class of RDE includes the MSV solution to the regime-switching model, but is larger than the set of MSV solutions since the RDE may also include a sunspot shock.

Using the representation (16) as our guide to specifying a perceived law of motion, we now turn to the stability of RDE under learning. Throughout, we assume that agents observe the current state  $s_t$  and know the true transition probabilities. This is consistent with the conventions of the adaptive learning literature that assumes agents observe contemporaneous exogenous variables, but not current values of endogenous variables.

Given that the CLDC is satisfied, the eigenvalues of  $(\oplus_{j=1}^m \beta_j) (P \otimes I_n)$  are inside the unit circle. Agents have a perceived law of motion (PLM) of the following form, which is consistent with the MSV solution,

$$y_t = A(s_t) + B(s_t)r_t \quad (17)$$

where  $A(j)$  is  $(n \times 1)$ , and  $B(j)$  is  $(n \times k)$ . Notice that we assume that agents do not know that in equilibrium the  $A_i = 0$ .<sup>11</sup>

Given the PLM in (17), expectations are state contingent, where  $s_t = j$  implies

$$E_t [y_{t+1} | s_t = j] = p_{j1}A(1) + p_{j2}A(2) + \dots + p_{jm}A(m) + \quad (18)$$

$$(p_{j1}B(1) + p_{j2}B(2) + \dots + p_{jm}B(m)) \rho r_t. \quad (19)$$

This produces a state-contingent ALM, or, equivalently, a state-contingent T-map

$$\begin{aligned} A(j) &\rightarrow \beta_j (p_{j1}A(1) + p_{j2}A(2) + \dots + p_{jm}A(m)) \\ B(j) &\rightarrow \beta_j (p_{j1}B(1) + p_{j2}B(2) + \dots + p_{jm}B(m)) \rho + \gamma_j. \end{aligned}$$

Conveniently, this state-contingent T-map may be stacked, and becomes the T-map associated to the stacked system under the PLM  $\hat{y}_t = A + Br_t$ , where, as before,  $B = (B(1)', \dots, B(m)')'$ , and also  $A = (A(1)', \dots, A(m)')'$ . The T-map is given by

$$T(A, B)' = ((\oplus_{j=1}^m \beta_j) (P \otimes I_n) A, (\oplus_{j=1}^m \beta_j) (P \otimes I_n) B \rho + \gamma),$$

and the RDE is a fixed point of  $T(A, B)$ . Here  $T : \mathbb{R}^{(nm \times 1)} \oplus \mathbb{R}^{(nm \times k)} \rightarrow \mathbb{R}^{(nm \times 1)} \oplus \mathbb{R}^{(nm \times k)}$ .

The eigenvalues of the Jacobian matrices

$$\begin{aligned} DT_A &= (\oplus_{j=1}^m \beta_j) (P \otimes I_n) \\ DT_B &= \rho' \otimes [(\oplus_{j=1}^m \beta_j) (P \otimes I_n)] \end{aligned}$$

govern E-stability. Thus, we obtain the following result:

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<sup>11</sup>In the univariate case below, we consider real time learning based on an alternative perceived law of motion of the following form

$$y_t = A + \hat{A}(s_t - 1) + Br_t + \hat{B}(s_t - 1)r_t,$$

where  $s_t$  acts as a dummy variable. While this formulation is more natural for real time learning, the E-stability results are identical in both cases and so we focus on the more parsimonious (17). As an additional alternative, agents can have a PLM with the same form as the stacked system.

**Proposition 2** *If the eigenvalues of  $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$  are inside the unit circle (i.e. the CLDC holds), then the unique RDE is E-stable.*

This result states that an economy described by the main expectational difference equation (4), with expectations formed using (18) and updated using least squares, will converge to the unique RDE.

#### 4.2.2 E-stability and Indeterminacy

Now we examine the stability of HDE, and again, for simplicity, and without loss of generality, we set  $\gamma_t = 0$ . We begin by considering VAR learning. In this case, the PLM takes the following form:

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t \quad (20)$$

where  $\xi_t$  is the m.d.s. sunspot variable independent of the Markov states. The PLM makes clear the primary distinction of HDE from the class of RDE solutions, since coefficients depend explicitly on  $s_t$  and  $s_{t-1}$ , whereas coefficients in the PLM for the RDE only depend on  $s_t$ .

Taking expectations conditional on the PLM given by (20) and values of  $(s_{t-1}, s_t)$  yields

$$E_t(y_{t+1}|s_{t-1} = i, s_t = j) = \sum_{k=1}^m p_{jk}A(j, k) + \left( \sum_{k=1}^m p_{jk}B(j, k) \right) (A(i, j) + B(i, j)y_{t-1} + C(i, j)\xi_t).$$

The T-map is given by

$$A(i, j) \rightarrow \beta_j \left[ \sum_{k=1}^m p_{jk}A(j, k) + \left( \sum_{k=1}^m p_{jk}B(j, k) \right) A(i, j) \right], \quad (21)$$

$$B(i, j) \rightarrow \beta_j \left( \sum_{k=1}^m p_{jk}B(j, k) \right) B(i, j), \quad (22)$$

$$C(i, j) \rightarrow \beta_j \left( \sum_{k=1}^m p_{jk}B(j, k) \right) C(i, j). \quad (23)$$

E-stability is determined by the Jacobian  $DT(\bar{A}, \bar{B}, \bar{C})$ , where  $\bar{A}, \bar{B}, \bar{C}$  are the HDE parameters found by solving (11)-(12). Given the complexity of the Jacobian, we

are not able to obtain general E-stability results for HDE. The next Section presents instability results for a univariate and New Keynesian example.

Now consider mean value learning. In this case, agents are assumed to understand the endogenous lagged dependence in the model and only estimate the state-dependent mean. Specifically, assume agents observe and understand the stochastic structure of the extrinsic noise process,  $\eta_t$ , where

$$\eta_t = \phi(s_{t-1}, s_t)\eta_{t-1} + \theta(s_{t-1}, s_t)\xi_t$$

for appropriately defined  $\phi(i, j), \theta(i, j)$ . Then we take our agent's forecasting model as

$$y_t = A(s_{t-1}, s_t) + B\eta_t$$

The corresponding T-map is

$$A(i, j) \rightarrow \beta_j \sum_{k=1}^m p_{jk} A(j, k)$$

and  $T(B) = B$ .<sup>12</sup>

**Proposition 3** *Assume HDE exist. If there exists a unique E-stable RDE, then the common factor representation of the HDE is E-stable.*

We note that under mean value learning, the CLDC governs stability of HDE.

## 5 Examples

The previous section provided the conditions under which an equilibrium is learnable, and demonstrated that in the case of RDE these conditions reduce to the CLDC, the uniqueness condition within the class of RDE. In this Section we illustrate the equilibrium selection results of this paper by examples. Using a simple univariate model, we consider HDE, its properties, and real time learning of RDE. In a New Keynesian example, we show that the HDE in an empirically realistic calibrated version of the model are not stable under learning when agents use a VAR based learning rule.

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<sup>12</sup>The result  $T(B) = B$  is standard in models with sunspots and reflects the fact that multiples of sunspots are also sunspots.

## 5.1 Univariate Model: RDE and HDE

To provide a concrete illustration of the class of solutions, we consider the special case where  $y_t$  is univariate,  $s_t$  takes values in  $\{1, 2\}$ , and  $\gamma_t = 0$ . Then

$$y_t = \beta_t E_t y_{t+1}. \quad (24)$$

Note that in this case, if there is a unique RDE, it is, trivially,  $y_t = y_{it} \Leftrightarrow s_t = i$ , where  $y_{it} = 0$  for  $i = 1, 2$ .

To compute an HDE, recall that a rational expectations equilibrium is a process  $y_t$  such that

$$y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t \quad (25)$$

where  $\xi_t$  is an m.d.s. that satisfies  $E_{t-1} \xi_t = 0$ . Of particular interest is the case in which “one regime is determinate and one regime is indeterminate,” or, formally, for example,  $|\beta_1| < 1 < \beta_2$ . In this case, non-degeneracy requires that regimes are not absorbing, so that  $p_{22} > 0$ . Define

$$\xi_t = \begin{cases} -\beta_1^{-1} y_{t-1} + \delta_{11} v_t & (s_{t-1}, s_t) = (1, 1) \\ \frac{P_{11}}{P_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1, 2) \\ -\beta_2^{-1} y_{t-1} + \delta_{21} v_t & (s_{t-1}, s_t) = (2, 1) \\ \frac{P_{21}}{P_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2, 2) \end{cases}$$

where  $\delta_{ij} \in \mathbb{R}$  is arbitrary, and  $v_t$  is any martingale difference sequence with uniformly bounded support. The dynamics for  $y_t$  follow

$$y_t = \begin{cases} \delta_{11} v_t & (s_{t-1}, s_t) = (1, 1) \\ \frac{1}{P_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1, 2) \\ \delta_{21} v_t & (s_{t-1}, s_t) = (2, 1) \\ \frac{1}{P_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2, 2) \end{cases}, \quad (26)$$

Note that provided  $|\beta_2 P_{22}| > 1$ ,  $y_t$  is UB. The process given by (26) is an HDE, since dynamics explicitly depend on  $s_t$  and  $s_{t-1}$ . Notice that the indeterminacy of region 2 spills over across regimes so that there is sunspot dependence in both regimes. It should be clear from this representation of an HDE that it is not possible to represent this class of equilibria in terms of a stacked system. In an RDE,  $y_t$  switches between two stochastic processes that are independent of the underlying Markov state. In an HDE the value of  $y_t$  depends on the current state  $s_t$  and also explicitly on the Markov state in the previous period. This dependence is self-fulfilling in the sense that it exists only because agents expect it.

We now turn to the stability of the univariate HDE given in (26), and we consider VAR learning. In this case, the parameters  $A, B, C$  in the PLM (20) are elements

of the real line. Computing conditional forecasts using this PLM, we obtain the following T-map for  $B$ :

$$B(i, j) \longrightarrow \beta_j (P_{j1}B(j, 1) + P_{j2}B(j, 2)) B(i, j) \quad (27)$$

Ignoring the boundedness requirement, a fixed point of this map identifies an HDE. The only restrictions, then, are the following:

$$1 = \beta_1 (P_{11}B(1, 1) + P_{12}B(1, 2)) = \beta_2 (P_{21}B(2, 1) + P_{22}B(2, 2)). \quad (28)$$

In particular, there is a two dimensional continuum of coefficients on lagged  $y$  providing fixed points.

Farmer, Waggoner, and Zha (2006) focus on particular fixed points, given by

$$B(1, 1) = B(2, 1) = 0, \quad B(1, 2) = \frac{\beta_1^{-1}}{P_{12}}, \quad B(2, 2) = \frac{\beta_2^{-1}}{P_{22}},$$

and  $d(i, j) = \delta_{ij}$ .

To analyze stability we compute the eigenvalues of  $DT$ . The first four equations decouple, and, when evaluated at the fixed point, provide the following Jacobian:

$$DT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta_1^{-1}\beta_2 P_{21}/P_{12} & \beta_1^{-1}\beta_2 P_{21}/P_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & P_{21}/P_{22} & 2 \end{pmatrix}.$$

The Jacobian has an eigenvalue of 2, which implies that under VAR learning, HDE are E-unstable.

This instability result for HDE generalizes a finding in the sunspots literature. The instability of HDE arise here because of the difficulty in coordinating on a particular sunspot. In an HDE, agents' expectations have the additional propagation that arises through the lagged endogenous state variable. Out of equilibrium, if agents' hold beliefs close to the equilibrium, when agents' extrapolate these beliefs the actual parameters move further away from the equilibrium. Hence, because the additional propagation only exists because of self-fulfilling expectations, they fail to coordinate on these sunspot equilibria. If, however, this serial propagation is exogenously imposed into agents' information set via the sunspot, then the equilibria can be stable.

## 5.2 Real Time Learning of RDE

The connection between E-stability and stability under real-time learning is made in constant parameter models by Evans and Honkapohja (2001). However, it is not

clear that the results in Evans and Honkapohja (2001) apply to the regime-switching framework. To address this issue, we present a real time learning formulation of regime dependent equilibria.

We again take  $y_t$  to be univariate, and assume  $s_t$  takes values in  $\{1, 2\}$ , but now we allow  $\gamma_t$  to be non-trivial. The model is given by

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t. \quad (29)$$

Assume  $(\beta_1 \oplus \beta_2)P$  has eigenvalues inside the unit circle, so that there is a unique RDE. To consider the stability under learning of this RDE, we provide agents with the following forecasting model

$$y_t = A + \hat{A}\hat{s}_t + Br_t + \hat{B}\hat{s}_t r_t,$$

where  $\hat{s}_t = s_t - 1$ . To simplify notation, let  $\theta = (A, \hat{A}, B, \hat{B})'$  and  $X = (1, \hat{s}_t, r_t, \hat{s}_t r_t)'$ . Agents estimate  $\theta$  by regressing  $y_t$  on  $X_t$ . Letting  $\theta_t$  be the time  $t$  estimate of  $\theta$ , the recursive formulation of this estimation procedure is given by

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} R_t^{-1} X_t (y_t - \theta'_{t-1} X_t) \\ R_t &= R_{t-1} + t^{-1} (X_t X_t' - R_{t-1}). \end{aligned} \quad (30)$$

The matrix  $R$  consists of the sample second moments of the regressors. The agents use these estimates, together with their forecasting model, to form expectations. These expectations are embedded into the expectational difference equation to obtain the actual law of motion and associated T-map, where the ALM may then be written  $y_t = T(\theta_{t-1})X_t$ . The T-map is given by

$$\begin{aligned} A &\rightarrow \beta_1(A + \hat{A}(1 - P_{11})) \\ \hat{A} &\rightarrow \beta_2(A + \hat{A}P_{22}) - \beta_1(A + \hat{A}(1 - P_{11})) \\ B &\rightarrow \beta_1(B + \hat{B}(1 - P_{11}))\rho \\ \hat{B} &\rightarrow \beta_2(B + \hat{B}P_{22})\rho - \beta_1(B + \hat{B}(1 - P_{11}))\rho. \end{aligned}$$

Imposing this into the algorithm (30) identifies a dynamic system that can be analyzed using the theory of stochastic recursive algorithms. Letting  $\theta^*$  be the fixed point of the T-map identifying an RDE, the learning question is, does  $\theta_t$  converge to  $\theta^*$  almost surely? We have the following proposition.

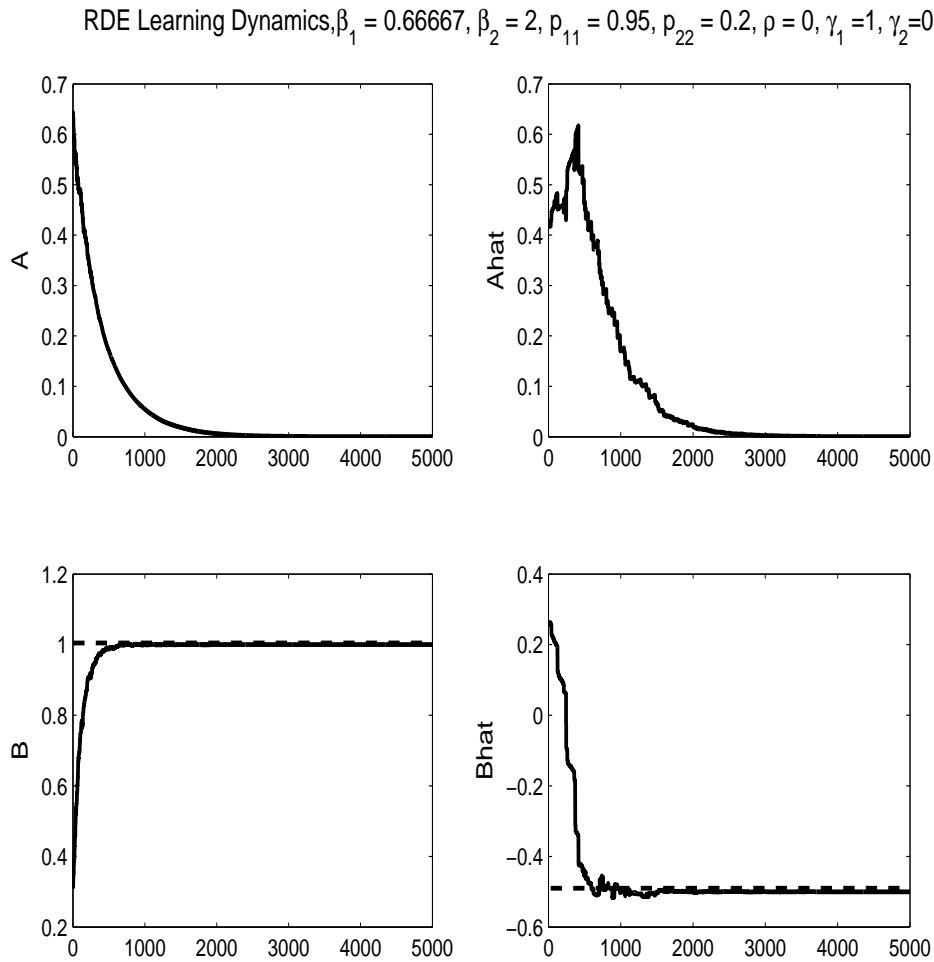
**Proposition 4** *If  $\gamma_t = 0$ , then, locally,  $\theta_t \rightarrow \theta^*$  almost surely.*<sup>13</sup>

<sup>13</sup>As is standard in the learning literature, in order to apply the theory of stochastic recursive algorithms requires imposing a ‘‘projection facility’’ on the recursive least squares algorithm. See Evans and Honkapohja (2001) for details.

The restriction  $\gamma_t$  is needed to simplify the proof, though we feel it is very likely that the proposition holds for  $\gamma_t \neq 0$ . The difficulty raised by non-zero  $\gamma$  reflects the fact that the state dynamics are no longer conditionally linear, a property that the convergence theorems typically rely upon.

To illustrate this result for  $\gamma \neq 0$ , we use simulations. We parameterize the model so that the CLDC is satisfied. This ensures the existence of a unique rational expectations equilibrium that is also an RDE. We set  $\beta_1 = 1/1.5, \beta_2 = 2, p_{11} = .95, p_{22} = .2, \rho = 0, \gamma_1 = 1, \gamma_2 = .5$ . For these parameter values the unique RDE coefficients are  $A_1 = A_2 = 0, B_1 = 1, B_2 = .5$ . We draw initial conditions for  $\theta$  randomly and simulate the model for 5000 time periods. Figure 1 plots a typical simulation. As the figure makes clear, the RDE is stable under least squares learning.

Figure 1: Real time learning of an RDE.



### 5.3 A New Keynesian Model

Farmer, Waggoner, and Zha (2007) illustrate the CLDC is necessary for determinacy, but not sufficient, and concludes that policymakers who focus on obeying the CLDC may not bring about a unique equilibrium. In this section, we use the parameter values from Farmer, Waggoner and Zha (2007) to construct a sunspot HDE. This example is of particular interest because the parameter values resemble estimates from Lubik and Schorfheide (2004), so are empirically plausible. The values reflect one view of monetary policy, namely, that the Federal Reserve did not adhere to the Taylor Principle during the 1970s, but did so subsequently. In this example, the CLDC holds and the RDE is unique, yet sunspot HDE also exist. Thus, whether the resulting HDE can arise in a setting where the CLDC holds and agents update their expectations using a reasonable learning algorithm, such as recursive least squares, is of particular interest.

The model, reproduced here for convenience, is given by

$$\begin{aligned}\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + g_t \\ x_t &= E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + u_t \\ i_t &= \alpha_t \pi_t + \gamma_t x_t,\end{aligned}$$

where

$$\alpha_t = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2 \end{cases}$$

and

$$\gamma_t = \begin{cases} \gamma_1 & \text{for } s_t = 1 \\ \gamma_2 & \text{for } s_t = 2 \end{cases}$$

The random variable  $s_t$  follows a finite-state Markov chain with transition probabilities  $p_{ij} \equiv \Pr [s_t = j | s_{t-1} = i]$  for  $i, j = 1, 2$ .

Parameters values are from Table 3 in Farmer, Waggoner, and Zha (2007) and are

$\beta$	$\sigma$	$\kappa$	$\alpha_1$	$\gamma_1$	$\alpha_2$	$\gamma_2$	$p_{11}$	$p_{22}$
.9949	1.655	.675	.77	.17	2.19	.30	.8577	.99

In this calibrated example, FWZ compute the HDE as<sup>14</sup>

$$\begin{aligned} c_1 &= 0.999795 \\ c_2 &= .738137 \\ v_1 &= \begin{pmatrix} -.977509 \\ -.210551 \end{pmatrix} \\ v_2 &= \begin{pmatrix} -0.010062 \\ 0.0065658 \end{pmatrix} \end{aligned}$$

where the reduced-form autocorrelation coefficients are computed by plugging into (10). It is straightforward to verify that the resulting stochastic process is uniformly bounded.

Under VAR learning, the PLM is given by (20), where Farmer, Waggoner, and Zha (2007) provide values for the coefficients for the HDE. Evaluating the T-map given by (21) – (23) at the above rational expectations parameter values leads to the Jacobian of the T-map relevant for E-stability. Because the  $DT_B$  block de-couples it is sufficient to examine only this portion of the Jacobian

$$\begin{bmatrix} DT_B^1 & DT_B^2 \\ DT_B^3 & DT_B^4 \end{bmatrix}$$

where  $DT_B^1, DT_B^4$  are given, respectively, by

$$\begin{bmatrix} p_{11}B(1,1)' \otimes \beta_1 + I \otimes \beta_1 (p_{11}B(1,1) + p_{12}B(1,2)) & p_{12}B(1,1)' \otimes \beta_1 \\ 0 & I \otimes \beta_2 (p_{21}B(2,1) + p_{22}B(2,2)) \end{bmatrix} \\ \begin{bmatrix} I \otimes \beta_1 (p_{11}B(1,1) + p_{1,2}B(1,2)) & 0 \\ p_{21}B(2,2)' \otimes \beta_2 & p_{22}B(2,2)' \otimes \beta_2 + I \otimes \beta_2 (p_{21}B(2,1) + p_{22}B(2,2)) \end{bmatrix},$$

and

$$DT_B^2 = \begin{bmatrix} 0 & 0 \\ p_{21}B(1,2)' \otimes \beta_2 & p_{22}B(1,2)' \otimes \beta_2 \end{bmatrix} \\ DT_B^3 = \begin{bmatrix} p_{11}B(2,1)' \otimes \beta_1 & p_{12}B(2,1)' \otimes \beta_1 \\ 0 & 0 \end{bmatrix}$$

Evaluating this Jacobian leads to repeated eigenvalues of 2.6323, 2.0755, .6635, .4197, 0. Therefore, the HDE are not E-stable based on VAR learning. However, we know from above that since the CLDC is satisfied, the RDE are E-stable, and the HDE are stable under mean value learning.

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<sup>14</sup>See Farmer, Waggoner, and Zha (2007) for details on the numerical procedure for computing these values.

## 6 Conclusion

This paper studies the existence and stability of two classes of rational expectations equilibria in a regime-switching rational expectations model under adaptive learning, extending the literature on learning to a non-linear framework. Building on the work of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2006, 2007), the two classes are:

- *Regime Dependent Equilibria*: An RDE is a uniformly bounded process that satisfies the regime-switching expectational difference equation and imposes the restriction that agents do not condition their expectations on lagged regimes (i.e. only the current regime enters the state vector).
- *History Dependent Equilibria*: An HDE is a process that satisfies the regime-switching expectational difference equation, where agents condition expectations on current and lagged values of the regime (i.e. current and past regimes enter the state vector).

The Conditionally Linear Determinacy Condition (CLDC), a generalization of the Long Run Taylor Principle of Davig and Leeper (2007), ensures the existence of a unique RDE that is also E-stable. When the CLDC is satisfied, there may still exist sunspot equilibria as demonstrated by Farmer, Waggoner, and Zha (2007). However, we demonstrate that in a univariate model and an empirically plausible New Keynesian model that these HDE may not be learnable, depending on the conditioning set imposed on boundedly rational agents.

Our results have implications for monetary policy design. As Farmer, Waggoner, and Zha (2007) demonstrate, establishing determinacy conditions for monetary policy across both class of equilibria may be elusive. These findings raise the question of whether determinacy and uniqueness of equilibria are the appropriate metrics when designing monetary policy. One reaction may be to abandon rule-based monetary policy, such as a Taylor rule, and to further embrace policy as solving complicated control problems. Instead, we argue that learnability is an important criterion for models with multiple equilibria and monetary policy should be designed so that there is a unique, expectationally-stable equilibrium. Our findings indicate that as long as monetary policy follows the Long Run Taylor Principle (i.e. the CLDC holds), then private agents that use adaptive learning to infer the mean and lag properties of the data will converge on the unique RDE, which is free of sunspot shocks.

## 7 Appendix

### Proof of Proposition 1

To establish part (b.), let  $y_{it}$  identify an RDE. Denote by  $f_t$  the time  $t$  density functions; for example,  $f_t(y, s|s_{t-1} = i, \Omega_{t-1})$  is the joint density of  $y_t$  and  $s_t$  conditional on  $s_{t-1} = i$  and on all other time  $t - 1$  information, not including current and past  $s_{t-1}$ , as captured by  $\Omega_{t-1}$ . Also, let  $f_t^i(y|\Omega_{t-1})$  be the density for  $y_{it}$  conditional on  $\Omega_{t-1}$ , and  $f(s|s_{t-1} = i)$  be the conditional density of  $s_t$  given  $s_{t-1} = i$  (course,  $f(s = j|s_{t-1} = i) = P_{ij}$ ). With this notation, we may compute expectations as follows:

$$\begin{aligned} E(y_{t+1}|s_t = i, \Omega_t) &= \int \int y f_{t+1}(y, s|s_t = i, \Omega_t) ds dy \\ &= \int \int y f_{t+1}(y|s, s_t = i, \Omega_t) f(s|s_t = i) ds dy \\ &= \int \int y f_{t+1}^s(y|\Omega_t) f(s|s_t = i) ds dy \\ &= \sum_j P_{ij} E_t y_{jt+1}, \end{aligned}$$

where the third equality precisely follows from the facts that  $y_t = y_{it} \Leftrightarrow s_t = i$  and that  $y_{it}$  is independent of  $s_{t+n}$  for all  $n$ . Now we may simply use this formula for the expectations of  $y_t$  to verify that the stacked system is satisfied.

The other parts of the proposition are established in the text in the Section on HDE.

### Proof of Proposition 4

Using the notation from the body of the paper, we may write the recursive algorithm as

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} S_{t-1}^{-1} X_t (y_t - \theta'_{t-1} X_t) \\ S_t &= S_{t-1} + t^{-1} (X_t X_t' - S_{t-1}) - \frac{1}{t^2} \frac{t}{t+1} (X_t X_t' - S_{t-1}), \end{aligned}$$

where  $X_t = (1, \hat{s}_t, r_t, \hat{s}_t r_t)$  and  $S_{t-1} = R_t$ . If  $X_t$  could be written as a linear difference equation in i.i.d. noise conditional on values of  $\theta$  and  $S$ , we could immediately apply the main results of the learning literature; however,  $X_t$  is not conditionally linear, so we must work harder: we must verify conditions  $M$  in Chapter 7.3 of Evans and Honkapohja (2001).

First, notice that the evolution of  $X_t$  is independent of  $\theta$  and  $S$ , simplifying our task. Let  $Q^n(x, \cdot)$  be the distribution of  $X_{t+n}$  given that  $X_t = x$ . We must demonstrate the following:

1. For all  $n, m$  there exists  $K$  so that  $\int (1 + \|y\|^m) Q^n(x, dy) \leq K (1 + \|x\|^m)$
2. For all  $p$  there exist  $K$  and  $\delta$  so that for all  $g \in \text{Li}(p)$ , for all  $n$  and for all  $x_1, x_2$ , we have

$$\left| \int g(y) Q^n(x_1, dy) - \int g(y) Q^n(x_2, dy) \right| \leq K \rho^n \|x_1 - x_2\| (1 + \|x_1\|^p + \|x_2\|^p)$$

Here  $\text{Li}(p)$  is a space of functions from  $\mathbb{R}^2$  to itself, defined in Evans and Honkapohja (2001): it turns out, as will be seen shortly, the simplicity of our set-up allows us to ignore the special properties of  $\text{Li}(p)$ , so we may simply take any  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

3. For all  $q \geq 1$  there exist  $n, \alpha < 1$  and  $\beta$  so that for all  $x$  we have

$$\int \|y\| Q^n(x, dy) \leq \alpha \|x\|^q + \beta.$$

The simplicity of our state dynamics allows these items to be easily demonstrated. Indeed,  $X_t$  is uniformly bounded a.s. by some number  $M$ , so  $\int (1 + \|y\|^m) Q^n(x, dy) \leq 1 + M^m$ , thus demonstrating item 1. Item 2, which would be quite difficult to demonstrate if  $\gamma_t \neq 0$  follows here because there only two states: the left-hand-side is

$$\begin{pmatrix} (1, 0) P^n \begin{pmatrix} g_1(x_1) \\ g_1(x_2) \end{pmatrix} \\ (0, 1) P^n \begin{pmatrix} g_2(x_1) \\ g_2(x_2) \end{pmatrix} \end{pmatrix},$$

which goes to zero exponentially because  $P$  is a stochastic matrix (here  $g_i$  is the  $i$ -th coordinate of  $g$ , and it is because there are only a finite number of states that we do not have to worry about the special properties of  $g$ ). Finally, item 3 follows in a fashion similar to item one because  $X_t$  is uniformly bounded.

Because the Markovian state dynamics satisfy the correct conditions, we may proceed as usual: stack the estimators  $S$  and  $\theta$  into a matrix  $\phi$  and write the recursive system as

$$\phi_t = \phi_{t-1} + \frac{1}{t} H(\phi_{t-1}, X_t) + \frac{1}{t^2} q(t, \phi_{t-1}, X_t). \quad (31)$$

The linearity of the T-map makes it straight-forward to verify that this recursion satisfies the necessary properties. Now set

$$h(\phi) = \lim_t E(H(\phi, X_t)).$$

The possible convergence points of (31) are the locally asymptotically stable fixed points of the differential equation  $\dot{\phi} = h(\phi)$ . Computing  $h(\phi)$  yields the decoupled

system

$$\begin{aligned}\frac{d\theta}{dt} &= S^{-1}E(X_t X_t')(T(\theta) - \theta) \\ \frac{dS}{dt} &= E(X_t X_t') - S.\end{aligned}$$

We conclude that locally asymptotic stability obtains provided the eigenvalues of  $DT$  have negative real part. The proof is completed by noting that the eigenvalues of  $DT$  are the eigenvalues of  $(\beta_1 \oplus \beta_2)P$ .

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