Implementing Optimal Monetary Policy in New-Keynesian Models with Inertia

George W. Evans
University of Oregon and University of St. Andrews

Bruce McGough
Oregon State University

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Abstract

We consider optimal monetary policy in New Keynesian models with inertia. First order conditions characterizing the unconditionally optimal rational expectations equilibrium (REE) are determined and are compared with those conditions identifying optimality from the timeless perspective. Implementation of optimal REE is considered via construction of interest-rate rules. An expectations based rule is derived that always yields a determinate model and an E-stable equilibrium. Further, the set of all interest-rate rules consistent with optimal REE is classified, and open subsets are shown to correspond to indeterminate models and unstable equilibria.

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1 Introduction

The canonical New-Keynesian model identifies two reduced form equilibrium restrictions in three endogenous variables: the equilibrium restrictions are commonly referred to as the New-Keynesian IS relation and the New-Keynesian Phillips curve (or AS relation), and the endogenous variables represent inflation, output gap, and the nominal short rate, which is often interpreted as the policy instrument. The model is closed and policy analysis is conducted by specifying a time-path for the instrument – either using a contingency plan or an additional sequence of reduced form restrictions – and then computing the associated equilibrium. Preferences over different equilibrium outcomes, and hence over different policy actions, may be modeled using a quadratic loss criterion, which is either taken as generic and so not tied to a particular specification of the economic model, or alternatively, derived as a second order approximation to average utility across private agents. This ordering of policy alternatives allows for the characterization and analysis of optimal monetary policy. For a detailed analysis of the New-Keynesian model, its variants, and derivations of the associated welfare functions, see Woodford (2003).

The equilibrium corresponding to optimal monetary policy may be identified by appending to the model a sequence of restrictions which correspond to the policy maker's first order conditions. Some authors, e.g. Svensson (2005) and Svensson and Woodford (2005), suggest interpreting these first order conditions as specific targeting rules. As discussed in Section 3.1, this interpretation assumes that policy makers can behave in some way not explicitly modeled so that the first order conditions are always satisfied. While this assumption is reasonable for theoretical results and stylized models, analyzing practical implementations of monetary policy requires a more explicit representation of central bank behavior. The alternative we favor is to model policy-maker behavior by specifying a contingency plan or interest-rate rule – referred to as a Taylor rule – that is consistent with the first order conditions' implementation.

Closing New-Keynesian models with Taylor rules – either rules specified to implement optimal policy or simple rules chosen without reference to a particular objective – has a long tradition in the monetary policy literature, and numerous specifications have been considered. Many studies in the literature have identified various objections to, and problems arising from, particular specifications. For example, McCallum and Nelson (2004) argue that implementable Taylor rules should condition on values that are observable to policy makers in real time, thus dismissing instrument rules that are contingent on current output gap and inflation. However, policies that condition on lagged variables or private sector expectations of future variables are known to have difficulties. Simple forward-looking interest-rate rules may yield indeterminate steady states even when they satisfy the Taylor principle, and associated to these steady states are multiple sunspot equilibria: see Evans and McGough (2005b) and Evans and McGough (2007).[1] Since, in the presence of multiple equilibria, the equilibrium on which agents ultimately coordinate may exhibit unwanted properties, indeterminacy is undesirable. Also, Bullard and Mitra (2002) find that under simple backward-looking instrument rules, even if the model has a unique equilibrium, it may fail to be stable under learning; thus instrument rules that impart determinacy may still result in bad outcomes as boundedly rational agents will be unable to coordinate on the unique rational expectations equilibrium. Finally, focusing on instrument rules designed to implement optimal monetary policy does not mitigate these issues: Evans and Honkapohja (2006) show that a backward-looking rule consistent with the timeless perspective may yield both indeterminacy and instability. From this brief discussion, and from the literature on which it draws, we arrive at three simple conclusions:

1. determinacy and stability under learning are independent notions;
2. implementable Taylor rules may result in indeterminacy and/or instability;
3. policy makers should constrain their attention to rules that impart both determinacy and equilibrium stability.

The literature just surveyed focuses on the purely forward-looking New Keynesian model; however, substantial levels of inertia in both the AS and IS relations are often present in applied models of the economy. Some authors have pointed out that the presence of inertia may mitigate indeterminacy issues by expanding the relative size of the determinacy region in policy space: see Levin, Wieland, and Williams (2003). On the other hand, as mentioned above, Bullard and Mitra (2002) find that inertia may exacerbate equilibrium

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[1] Even if the steady state is determinate the model may still support multiple equilibria: see Benhabib and Eusepi (2005) for details.

instability. These observations raise the natural question: how does inertia affect the performance monetary policy as implemented by Taylor rules? In this paper, we address this question by studying the behavior of Taylor rules designed to implement optimal policy in New Keynesian models with inertia.[2]

Identification of the solution to the government's optimization problem depends on the surprisingly subtle definition of "optimality." The subtlety arises because, absent a commitment technology, the fully optimal rule is not time-consistent. To address this issue, Woodford (2003) advises the *timeless perspective*, which requires that policy makers internalize the cost of exploiting private agents expectations: in effect, the policy maker implements the policy she would have chosen in the distant past. The timeless perspective manifests itself as a simple time-invariant linear restriction: for a characterization of policy that is optimal from the timeless perspective, see Giannoni and Woodford (2002b).

An alternative to the timeless perspective has been proposed by Jensen and McCallum (2002) and Blake (2002). These authors suggest interpreting the commitment technology available to the government as adherence to a linear, time-invariant restriction analogous in functional form to the restriction identified by timeless perspective. Noting that under certain conditions a reasonable policy implements an asymptotically stationary economy, Jensen and McCallum (2002), and separately, Blake (2002) further suggest evaluating various policies by computing the expected value of the government's criterion across initial conditions, where the initial conditions are drawn from the associated asymptotic distribution; in this way, these authors are seeking time invariant targeting rules that implement an equilibrium that is, in a natural sense, "unconditionally optimal." In a purely forward-looking model, Jensen and McCallum (2002) and Blake (2002) characterize the unconditionally optimal REE. Using a calibrated model and the unconditional expected value of the policy-maker's objective as a metric, Jensen and McCallum (2002) find that the unconditionally optimal REE provides a 10.6% improvement over the timeless perspective.[3]

We begin our analysis of optimal monetary policy implementation in New Keynesian models with inertia by characterizing the unconditionally optimal

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[2] In Evans and McGough (2005a), we conducted a similar analysis of simple Taylor rules in models with inertia.

[3] It could be argued that the quantitative comparison is not really fair because the timeless perspective is chosen using a criterion that penalizes the exploitation of private sector expectations.

REE and the REE that is optimal from the timeless perspective. Following Jensen and McCallum (2002), we then measure the improvement provided by the unconditionally optimal REE over the timeless perspective across some calibrations of the model's parameters. We find that while for some calibrations the improvement over the timeless perspective is insignificant, for other calibrations, and particularly when the government places priority on output stabilization, the improvement may be high – as high as 17% for calibrated models and much higher in case of strong serial correlation in the exogenous shocks. Interestingly, the presence of inertia in the model tends to mitigate this improvement, though the relationship between the degree of inertia and the level of mitigation is complex and non-monotonic.

Having identified the optimal REE of interest, we turn to policy implementation: to implement their monetary policy, we impose that policy makers specify an interest-rate rule consistent with the optimal REE. Consistence with the optimal REE does not uniquely identify an interest-rate rule; indeed, it is well-known that a given equilibrium may be consistent with multiple interest-rate rules: see Evans and Honkapohja (2006). To proceed with our policy analysis, we characterize the collection of all Taylor rules within a certain linear class that are capable of implementing a given optimal REE: we call this collection of Taylor rules the "optimal policy manifold."

As suggested by the above literature review, not all interest-rate rules within the optimal policy manifold are created equal: some may generate indeterminacy, instability or both. In fact, it is not obvious that there exists a rule within the manifold that imparts both stability and determinacy. Some results are known: in a purely forward-looking model, Evans and Honkapohja (2006) derive an interest-rate rule (which we call the "EH-rule") exhibiting dependence on agents' expectations, as well as on lagged endogenous variables and current shocks, which is consistent with the timeless perspective, and which always yields a stable and determinate steady state. We generalize this rule by deriving it for both the timeless perspective and UO (unconditionally optimal) policy when inertia is present in the model, and show that the resulting model is always determinate. While analytic results for E-stability are not available, we find numerically that for all calibrations and inertial specifications considered in our paper, as well as for both the timeless perspective and UO-policy, the generalized EH-rule yields an E-stable equilibrium.

Analytic results are not available for arbitrary rules within the optimal policy manifold, but numerical analysis is possible. We find that open regions of the manifold correspond to instability, indeterminacy, or both, as well as to the presence of sunspot equilibria that are stable under learning. We also show that this finding serves as a caution to policymakers interested in employing rules, like the generalized EH-rule that are dependent on values of structural parameters: the location of the generalized EH-rule may be near a region of indeterminacy or instability; precise knowledge of structural parameters may be required to avoid bad outcomes.

This paper is organized as follows: Section 2 reviews the model and theory needed to obtain and discuss the paper's main points; Section 3 obtains, in the New Keynesian model with inertia, the first-order conditions necessary to implement optimal policy from the timeless perspective, and presents the proposition describing UO-policy; Section 4 focuses on implementable policy in the form of interest-rate reaction functions that are consistent with the timeless perspective or the UO-policy, and then examines the stability and determinacy properties of the New Keynesian model closed with these policy rules; and finally, Section 5 concludes.

2 The New Keynesian Model

We study a hybrid version of the New Keynesian monetary model as given by

$$\begin{aligned} IS: x_t &= -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1-\delta)x_{t-1} + g_t && (1) \\ AS: \pi_t &= \beta(\gamma E_t\pi_{t+1} + (1-\gamma)\pi_{t-1}) + \lambda x_t + u_t. && (2) \end{aligned}$$

Here x_t is the proportional output gap, π_t is the inflation rate, and g_t and u_t are independent, exogenous, stationary, zero-mean AR(1) shocks with damping parameters $0 \leq \rho_g < 1$ and $0 \leq \rho_u < 1$ respectively.

The first equation is a formulation of the forward-looking IS curve amended to include inertia: see e.g. Smets (2003). The second equation is the forward-looking Phillips curve. When $\gamma = 1$, equation (2) is the purely forward-looking New Keynesian "AS" relationship based on "Calvo pricing," and employed in Clarida, Gali, and Gertler (1999) and Ch. 3 of Woodford (2003).[4] The specification of the AS curve in the case $0 < \gamma < 1$ incorporates an inertial term and is similar in spirit to Fuhrer and Moore (1995), the Section 4 model of Gali and Gertler (1999), and the Ch. 3, Section 3.2

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[4] For the version with mark-up shocks see Woodford (2003) Chapter 6, Section 4.6.

model of Woodford (2003), each of which allows for some backward looking elements. Models with $0 < \gamma < 1$ are often called "hybrid" models, and we remark that in some versions, such as Fuhrer and Moore (1995), $\beta = 1$, so that the forward and backward looking components sum to one, while in other versions $\beta < 1$ is possible.

The model may be closed by specifying monetary policy, and we consider alternate optimal specifications in Section 3. Once a policy has been specified, the model may be placed in reduced form and the number and nature of its equilibria may be analyzed. As is standard, we say that the model is "determinate" if there is a unique nonexplosive REE, "indeterminate" if there are many such REE, and "explosive" otherwise. Methods for assessing the determinacy properties of a model are well known and we refrain from discussing them here: see Evans and McGough (2005b) for a detailed analysis of determinacy in New Keynesian models.

Whereas the determinacy properties of the model identify the number of equilibria present, the nature of these equilibria – whether they are stable under learning – is determined using the notion of expectational stability, which we briefly review now. An REE is stable under learning provided that if agents estimate the parameters of a forecasting model using least squares regression and form their expectations accordingly, then the economy will eventually converge to the equilibrium. Because the models are self-referential, that is, because the evolution of the economy depends on how agents form expectations, the stability of an REE under least squares learning cannot be taken for granted.

Analysis of stability under learning is usually conducted using expectational stability (E-stability). This is because, for a wide range of models and solutions, E-stability has been shown to govern the local stability of REE under least squares learning. In many cases this correspondence can be proved, and in cases where it cannot be formally demonstrated the "E-stability principle" has been validated through simulations. For a thorough discussion of E-stability see Evans and Honkapohja (2001).

The E-stability technique is based on replacing the stochastic algorithm capturing the evolution of agents' forecasting model estimates with a stylized learning mechanism based on a mapping which takes the agents' parameterized perceptions of how the economy evolves over time to the "truth" as captured by the economy's true data generating process given agents' perceptions. A fixed point of this map identifies the alignment of agents' perceptions with the truth, and therefore represents a rational expectations

Table 1: Calibrations

Name	ϕ	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3
MJ	.164	.02

Table 2: Inertial Specifications

Name	γ	δ
Forward	1	1
Small Lag	.75	.75
Medium Lag	.5	.5
Large Lag	.25	.25
Full Lag	.01	.01

equilibrium. The stylized learning mechanism is implemented by assuming that agents update their perceptions by moving them in the direction of the truth. If this mechanism leads perceptions to the fixed point of the map, then the associated equilibrium is said to be E-stable.

Some analytic results on stability and determinacy are available, but numeric methods must be used in the general case, and this requires assigning values to the model's parameters. We consider four calibrations of the parameters in the IS-AS curves, as due to Woodford (1999b), Clarida, Gali, and Gertler (2000), McCallum and Nelson (1999), and Jensen and McCallum (2002): the relevant parameter values are given in Table 1.

With these calibrations, we consider five inertial specifications, as given by Table 2. Finally, for each calibration and inertial specification, we will consider discount values of $\beta = .98$, $\beta = .99$ and $\beta = 1$.

3 Optimal Policy

We model the preferences of policymakers over alternate equilibrium paths as given by

$$E_t\left(\sum_{k=0}^{\infty}\beta^k(\psi x_{t+k}^2+\pi_{t+k}^2)\right). \tag{3}$$

For some specifications of the New Keynesian model, a loss criterion of this form may be taken as capturing a second order approximation to aggregate private agent utility, in which case ψ is a function of the model's deep parameters.[5] Rather than fixing the link between government preferences and the underlying model, we take (3) as a generic loss function parameterized by ψ, which simply captures the relative importance of output stabilization over inflation control. Note we have assumed, for simplicity, that the government and private agents have the same discount factor.

To construct optimal monetary policy, we begin by identifying the optimal REE, that is the REE which minimizes the government's loss subject to the Phillips curve constraint. In this sense, we are implicitly assuming π_t to be the government's instrument: see Section 3.1 for more discussion. This assumption makes the IS relation superfluous, and the associated reduced form model is obtained by combining aggregate supply with a first order restriction characterizing government behavior. The corresponding time-path of output and inflation represents the optimal REE consistent with the government's objective and the AS-relation. In Section 4, we will use the IS relation to establish the time-path of the interest rate consistent with this optimal REE. Interpreting this time-path as an interest-rate rule will then operationalize our optimal policy.

The optimal REE may be identified using Lagrange's method. Let L_t be the time t conditional Lagrangian. Exploiting the law of iterated expectations yields

$$L_t = E_t\left(\sum_{k=0}^{\infty}\beta^k\left(l(x_{t+k},\pi_{t+k})+\omega_{t+k}\left(\pi_{t+k}-R(x_{t+k},\pi_{t+k+1},\pi_{t+k-1})\right)\right)\right) \tag{4}$$

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[5] The loss criterion associated to models with inertia may include a dependence on lagged endogenous variables.

where

$$\begin{aligned} l(x_t, \pi_t) &= -(\psi x_t^2 + \pi_t^2) \\ R(x_t, \pi_{t+1}, \pi_{t-1}) &= \beta\gamma\pi_{t+1} + \beta(1-\gamma)\pi_{t-1} + \lambda x_t + u_t \end{aligned}$$

The associated FOC are given by

$$\omega_{t+k} = \frac{2\psi}{\lambda} x_{t+k} \tag{5}$$

$$2\pi_{t+k} = -\omega_{t+k} + \gamma\omega_{t+k-1} + \beta^2(1-\gamma)E_{t+k}\omega_{t+k+1} \tag{6}$$

Because the policy maker is not obligated to meet the expectations of the time -1 agents, the fully optimal solution takes $\omega_{-1} = 0$. This solution is not time consistent because it instructs the policy maker to behave differently in the initial period than in subsequent periods.

3.1 The Timeless Perspective

To address the inconsistency problem, Woodford (1999a) advises the timeless perspective, which dictates that policy makers modify their optimization problem by taking ω_{-1} to capture the cost to violating past commitments: $\omega_{-1} = \frac{2\psi}{\lambda}x_{-1}$. Equations (5) and (6) may then be combined to obtain the following relation between current inflation and current, lagged and expected future output gap:

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \gamma x_{t-1} - \beta^2(1-\gamma)E_t x_{t+1}\right). \tag{7}$$

For $\gamma = 1$, equation (7) reduces to the standard first order condition in the case of no inertia.

Whereas fully optimal policy, obtained by setting $\omega_{-1} = 0$, is not internally consistent, policy described by (7) is consistent not only in that it is time-invariant, but also, in the words of Woodford (1999a), this policy describes a pattern of behavior " ...which (the policy maker) would have wished to commit itself to at a date far in the past, contingent upon the random events that have occurred in the meantime." Therefore, by incorporating the restriction on the time -1 constraint multiplier, policy makers face an optimization problem designed to internalize the cost of exploiting private agents expectations. This anchors expectations and results in a policy that avoids the pitfalls of both inflation bias and stabilization bias.[6]

Until now we have assumed, somewhat unrealistically, that inflation is the government's policy instrument, and that it can be set to satisfy (7). Indeed, some authors suggest taking the first order condition (7) as a specific targeting rule. Interpreted this way, and assuming policymakers can, in some way not explicitly modeled, impose that (7) holds at every point in time, (2) and (7) comprise a fully specified reduced form model. Furthermore, arguments in Woodford (2003) guarantee this model to be determinate, and hence yield a unique rational expectations equilibrium.[7] Whether the unique REE is stable under learning must be determined numerically. We find that for all calibrations and inertial specifications, and with all discount factors, expectational stability always obtains.

Because the specific targeting rule interpretation does not provide a model of policy-maker behavior, some ambiguity remains concerning precisely how monetary policy is implemented. We address this issue in Section 4. There we design interest-rate reaction functions consistent with the optimal REE defined by the timeless perspective or unconditionally optimal policy (defined below). However, like Evans and Honkapohja (2006), we find that such policy rules do not necessarily imply (7) is satisfied, and so the issues of determinacy and stability must be revisited.[8] Because of this, we recommend choosing instrument rules consistent with (7) that also result in a determinate model with an E-stable equilibrium.

3.2 Unconditionally optimal policy

While the timeless perspective has a number of appealing qualities, including internal consistency, there is a continuum of time-invariant restrictions of

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[6] The timeless perspective has a number of additional advantages. Benigno and Woodford (2008) show that by appropriately modifying the government's valuation function, the timeless perspective may be viewed as the solution to a recursive programming problem; the methods they use are similar to the recursive saddle path techniques developed by Marcet and Marimon (1998). Benigno and Woodford also show that the modified recursive problem used to capture the timeless perspective is well-suited to linear-quadratic approximations. For more on the timeless perspective, see Woodford (2003), Giannoni and Woodford (2002a). Giannoni and Woodford (2002b) and McCallum (2005).

[7] See Footnote 8 on page 542.

[8] Policy rules consistent with the timeless perspective's first order condition may impart indeterminacy and some of the associated equilibria will not satisfy (7).

the form (7); and, since the timeless perspective is not fully optimal when measured against the objective (3), it is natural to wonder whether there are time-invariate restrictions that outperform the timeless perspective, at least on average, where the average is taken across initial conditions drawn with respect to their asymptotic distribution. Jensen and McCallum (2002) and Blake (2002) considered precisely this question in a purely forward looking model. Both Jensen and McCallum and Blake show numerically that the relationship

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \beta x_{t-1}\right) \tag{8}$$

is superior to the timeless perspective: Jensen and McCallum simply state it as an example of an improvement; but Blake proceeds to show that in fact (8) is the optimal rule of that form. Following Damjanovic, Damjanovic, and Nolan (2008), we call policy implementing the FOC (8) "unconditionally optimal policy" (UO-policy).

UO-policy is appealing in that it is time-invariant, and, on average, it minimizes (3). However, unlike the timeless perspective, it is not internally consistent, and therefore its implementation requires a stronger form of commitment technology.[9,10] Finally, while Jensen and McCallum (2002) obtain a maximum improvement of 10.26% over the timeless perspective, McCallum and Nelson (2004) note that in general, the average welfare improvements imparted by unconditionally optimal policy are not large. In the sequel we refrain from taking a stand on the superiority of either optimal policy condition: we consider both the timeless perspective and unconditionally optimal policy with the primary goal of analyzing their implementation via instrument rules.

To determine the first order condition corresponding to unconditionally optimal policy in the case of model inertia, we use the technique developed in Damjanovic, Damjanovic, and Nolan (2008). Acting on the time t conditional Lagrangian L_t, (4), by the unconditional expectations operator, multiplied

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[9] We think of the commitment technology required for implementation of the timeless perspective as being the willingness of the bank to accept the additional multiplier constraint.

[10] Also, Benigno and Woodford (2008) point out that "uniformly" optimal policy "... has the unappealing feature of giving a rule that leads to different long-run average values of an endogenous variable (e.g. the capital stock) "credit" for a higher initial average value of the variable as well." See footnote 66, page 44.

by $(1-\beta)$, we obtain:

$$\begin{aligned}
& (1-\beta)EL_t \\
= \;& (1-\beta)E\left(\sum_{k=0}^{\infty}\beta^k\left(l(x_{t+k},\pi_{t+k})+\omega_{t+k}\left(\pi_{t+k}-R(x_{t+k},\pi_{t+k+1},\pi_{t+k-1})\right)\right)\right) \\
= \;& E\left(-\psi x_t^2-\pi_t^2+\omega_t\left(\pi_t-\beta\gamma\pi_{t+1}-\beta(1-\gamma)\pi_{t-1}-\lambda x_t-u_t\right)\right) \\
= \;& E\left(-\psi x_t^2-\pi_t^2+\omega_t\pi_t-\beta\gamma\omega_t\pi_{t+1}-\beta(1-\gamma)\omega_t\pi_{t-1}-\lambda\omega_t x_t-\omega_t u_t\right) \\
= \;& E\left(-\psi x_t^2-\pi_t^2+\omega_t\pi_t-\beta\gamma\omega_{t-1}\pi_t-\beta(1-\gamma)\omega_{t+1}\pi_t-\lambda\omega_t x_t-\omega_t u_t\right),
\end{aligned}$$

where the second and last equalities are obtained by exploiting the stationarity of the various processes: $E(\omega_t\pi_{t+1})=E(\omega_{t-1}\pi_t)$, etc. This manipulation highlights the key difference between the timeless perspective and unconditional optimal policy: the former takes ω_{t-1} as given whereas the latter accounts for its average impact. Differentiating this last expression with respect to π_t and x_t gives the FOC, which we summarize in the following proposition:[11]

Proposition 1 *The unconditionally optimal policy for the model determined by (3) and (2) is given by*

$$\pi_t=-\frac{\psi}{\lambda}\left(x_t-\beta\gamma x_{t-1}-\beta(1-\gamma)E_t x_{t+1}\right). \tag{9}$$

The reduced form model given by (2) and (9) is necessarily determinate, and we call its unique bounded solution the "unconditionally optimal REE." Analytical results for E-stability of this REE are not available, but we find numerically that for all calibrations and inertial specifications, it is stable under learning.[12]

Notice that in the case $\gamma=1$ so that there is no inertia in the Phillips curve, (9) reduces to (8), the condition suggested by McCallum and Jensen, and established as optimal by Blake. Notice, too, that for $\beta=1$, (9) and the timeless perspective are identical as expected.

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[11] We obtained the FOC identified in this proposition independently, before we were aware of the work of Damjanovic, Damjanovic, and Nolan (2008). Because our proof is complicated and applies only to the hybrid New Keynesian model, we omit it, and instead rely on their method.

[12] Here were are interpreting (9) as a specific targeting rule, and again, concerns regarding its implementation apply. We address these concerns by constructing explicit interest-rate rules in Section 4.

3.3 Comparing the Timeless Perspective and the UO-policy

The results of the previous section indicate that the UO-policy (9) at least weakly dominates the timeless perspective (7) when average government loss is used as the comparison criterion; however, UO-policy is simply a small continuous transformation of the timeless perspective, so it is quite natural to wonder how much improvement is obtained. This question has been considered by Jensen and McCallum (2002) in the non-inertial case, with $\lambda = .02$, for varying values of ψ, β, and ρ. They obtain a maximum improvement of 10.26%. We conducted a similar analysis, and some of the results we obtained are reported in Table 3.

As expected, the UO-policy always yields an improvement over the timeless perspective; however, as noted by McCallum and Nelson (2004), the improvement is typically quite small: only in the forward-looking case $\gamma = 1$, and with heavy weight place on output stabilization, is the improvement significant. Interestingly, the presence of inertia seems to mitigate the second-best nature of the timeless perspective: for a hybrid model with $\gamma = .5$, the maximum improvement across calibrations is only .06%. Further examination of this mitigating effect indicates a complex, non-monotonic pattern. Consider Figure 1, which plots the percent improvement of the UO-policy over the timeless perspective for the Woodford calibration and varying γ. As γ increases to .5, which corresponds to the Medium Lag inertial specification, the improvement drops from 1.5% to near zero before rising again. This pattern is qualitatively the same as seen across calibrations, with the minimum improvement obtaining near $\gamma = .5$.

Fig 1 here

The magnitude of the serial correlation in the markup shock also impacts the improvement of UO-policy over the timeless perspective, with the numerically obtained qualitative result being that for most calibrations and inertia specifications, larger serial correlation leads to larger improvement; however, for inertial specifications with γ near .5, the relationship between serial correlation and improvement is again non-monotonic: see Figure 2 as an example.

Fig 2 here

Table 3: TP/UO Comparison (Forward Model and Medium Lag, $\beta = .98$)

	Forward Model				Medium Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	2.42%	.45%	.04%	3.08%	.04%	.01%	.00%	.04%
1	8.76%	2.46%	.30%	10.16%	.06%	.04%	.01%	.05%
10	16.12%	8.81%	1.79%	17.08%	.01%	.05%	.03%	.01%

4 Implementing Optimal Policy

We have characterized the optimal REE via first order conditions, taking the AS relation as a constraint. Our goal now is to investigate how the optimal REE can be implemented. That is, we look for interest-rate reaction functions that, when combined with the AS and IS relations, result in the optimal REE attaining according to determinacy and E-stability. To this end, we "nest" the timeless perspective and UO-policy in the following generalized FOC:

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \xi\gamma x_{t-1} - \frac{\beta^2}{\xi}(1-\gamma)E_t x_{t+1}\right), \tag{10}$$

where $\xi = 1$ in the case of the timeless perspective and $\xi = \beta$ in the case of UO-policy.

4.1 The Fundamentals Rule

As noted above, the system of expectational difference equations (2) and (10) is determinate for either value of ξ. Letting $y = (x, \pi)'$, the unique stationary equilibrium can be written

$$y_t = by_{t-1} + cu_t. \tag{11}$$

Now recall the IS-relation (1), repeated here for convenience:

$$x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1-\delta)x_{t-1} + g_t. \tag{1}$$

To create interest-rate rules consistent with the optimal REE, we follow Evans and Honkapohja (2006) (EH) and suppose first that agents have RE with expectations given by (11). Imposing these expectations into the IS relation (1), we can then solve for i_t, thus obtaining an interest-rate rule of the form

$$i_t = \tilde{\alpha}^L y_{t-1} + \tilde{\alpha}^{\hat{g}} \hat{g}_t, \tag{12}$$

where $\hat{g} = (g, u)'$. We call (12) the "fundamentals rule." The values of the 1×2 matrices $\tilde{\alpha}^L$ and $\tilde{\alpha}^{\hat{g}}$ depend on the model's reduced form parameters as well as which FOC is used.

We may now consider the full reduced form model, assuming monetary policy is implemented by following the fundamentals rule. The model is given by (2), (1), and (12). Results concerning a special case of this model are known. Evans and Honkapohja (2006) obtained the fundamentals rule in the case that $\gamma = \delta = 1$ and the timeless perspective is assumed. They showed analytically that the model may be indeterminate and that the equilibria are always unstable under learning. Similarly, we find

Proposition 2 *In the case $\gamma = \delta = 1$ and UO-policy is used, under the fundamentals based rule, the economy may be indeterminate, and the equilibria are always unstable under learning.*

The proof of this proposition mimics the proof by EH, and we suppress the details.

Similar work can be done in the case of inertia; however, we must proceed numerically as analytic results are unavailable. Some of our results are collected in Table 4. Here $\beta = .99$ and UO-policy is used. To identify the stability and determinacy properties, we use the notation SD (stable determinacy), UD (unstable determinacy), SI (stable indeterminacy, that is, E-stable sunspot equilibria), and UI (unstable indeterminacy).[13] Similar results obtain for other calibrations and inertial specifications and for the timeless perspective.

Interpreting the first order conditions given by the timeless perspective or UO-policy as specific targeting rules, and assuming that policymakers are in some unspecified way able to achieve the specific targeting rule so that only the AS relation is considered as a restriction, both the timeless perspective and UO-policy resulted in stable determinacy. The work of Evans and Honkapohja (2006) and the results here show that simply assuming rational

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[13] The analysis of sunspot stability raises the issue of equilibrium representation. When we say that there exist E-stable sunspot equilibria, we will always mean that there exist sunspot equilibria with at least one stable representation. For details on representations, see Evans and McGough (2005b).

Table 4: Determinacy and E-stability of the Fundamentals Rule

	Small Lag				Medium Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	UI	UI	SD	UI	UI	UI	SD	SD
1	UD	UD	UI	UI	UI	UI	SD	UI
10	UD	UD	UI	UI	UI	UD	UI	UI

	Large Lag				Full Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	SD	UI	SD	SD	SD	SD	SD	SD
1	UI	UI	SD	SD	SD	SD	SD	SD
10	UI	UI	SD	SD	SD	SD	SD	SD

expectations (RE) and using the IS relation to design an interest-rate rule consistent with the associated optimality condition can be destabilizing and hence ill-advised.

4.2 The EH-rule

The results reported in Proposition 2 and Tables 4 and **??** warn against the use of the fundamentals rule. Evans and Honkapohja, in the case $\gamma = 1$ and under the timeless perspective, faced a similar problem, and proposed the following solution. Instead of forming a rule by first computing expectations under RE, EH suggest taking expectations as given. Specifically, combine the general FOC (10) and the Phillips curve (2) and solve for x_t to obtain

$$x_t = \frac{\lambda}{\psi + \lambda^2}\left(\frac{\psi\xi\gamma}{\lambda}x_{t-1} + \frac{\psi\beta^2(1-\gamma)}{\lambda\xi}E_t x_{t+1} - \beta\gamma E_t\pi_{t+1} - \beta(1-\gamma)\pi_{t-1} - u_t\right).$$

This equation may then be combined with the IS relation (1) to obtain a rule of the form

$$i_t = \hat{\alpha}^f E_t y_{t+1} + \hat{\alpha}^L y_{t-1} + \hat{\alpha}^{\hat{g}}\hat{g}_t \tag{13}$$

where

$$\hat{\alpha}^f = \left(\frac{\delta\xi(\psi+\lambda^2)-\psi\beta^2(1-\gamma)}{\xi\phi(\psi+\lambda^2)}, 1+\frac{\lambda\beta\gamma}{\phi(\psi+\lambda^2)}\right) \tag{14}$$

$$\hat{\alpha}^L = \left(\frac{(1-\delta)(\psi+\lambda^2)-\psi\gamma\xi}{\phi(\psi+\lambda^2)}, \frac{\lambda\beta(1-\gamma)}{\phi(\psi+\lambda^2)}\right) \tag{15}$$

$$\hat{\alpha}^{\hat{g}} = \left(\frac{1}{\phi}, \frac{\lambda}{\phi(\psi+\lambda^2)}\right). \tag{16}$$

We call this the EH-rule and note that in the case $\gamma = 1$, $\delta = 1$ and the timeless perspective is assumed, (13) reduces to the rule obtained by Evans and Honkapohja.

Evans and Honkapohja showed analytically that their rule always resulted in determinacy, and that the unique REE was always stable under learning. Similarly,

Proposition 3 *The reduced form model, given by the IS and AS relations (1) and (2), and closed with the EH-rule (13), is determinate for both $\xi = 1$ and $\xi = \beta$.*

The proof of this proposition follows from the observation that solutions to the system (1), (2) and (13) are bijective with those of (2) and the relevant FOC. This proposition extends the determinacy result of Evans and Honkapohja to models with inertia, and to rules implementing either the timeless perspective or UO-policy. Analytic results on stability are not obtainable for us, but numerically we find the same result as EH. In particular, for all permutations of calibrations and inertial specifications, and under both the timeless perspective and UO-policy, the EH-rule results in stability.

4.3 The Optimal Policy Manifold

Equations (12) and (13) indicate that there are at least two rules of the form

$$i_t = \alpha^f E_t y_{t+1} + \alpha^L y_{t-1} + \alpha^{\hat{g}} \hat{g}_t \tag{17}$$

consistent with the optimal REE (11), and further show that rules implementing the optimal REE may not impart the same stability and determinacy properties. We are then led to wonder if the collection of all rules of the form

(17) can be classified, and how their associated stability and determinacy properties may vary. We turn to these issues now.

Denote by Ω the set of all $\alpha = (\alpha^f, \alpha^L, \alpha^{\hat{g}})'$ such that (17) is consistent with (11). We call Ω the optimal policy manifold, and note that it is a subset of $\mathbb{R}^6$, and furthermore is non-empty, as it contains both the fundamentals rule (12) and the EH-rule (13). Now define the matrix $\hat{c} = (0, c)$, so that the optimal REE may be written $y_t = by_{t-1} + \hat{c}\hat{g}_t$. Forming expectations with respect to this REE and imposing them into (17) yields

$$i_t = (\alpha^f b^2 + \alpha^L)y_{t-1} + (\alpha^{\hat{g}} + \alpha^f (b\hat{c} + \hat{c}\rho)\, \hat{g}_t,$$

which must be the same as the fundamentals rule in an REE. Thus $\alpha \in \Omega$ provided

$$\begin{aligned} \alpha^f b^2 + \alpha^L &= \tilde{\alpha}^L & (18) \\ \alpha^f (b\hat{c} + \hat{c}\rho) + \alpha^{\hat{g}} &= \tilde{\alpha}^{\hat{g}}. & (19) \end{aligned}$$

Equations (18) and (19) characterize the optimal policy manifold. Furthermore, notice that we may trivially solve (18) and (19) for α^L and $\alpha^{\hat{g}}$ as linear functions of α^f. Thus the optimal policy manifold is a two-dimensional hyperplane in $\mathbb{R}^6$, given by

$$\Omega = \left\{\alpha \in \mathbb{R}^6 : \exists \alpha^f \in \mathbb{R}^2 \text{ with } \alpha^L = \tilde{\alpha}^L - \alpha^f b^2 \text{ and } \alpha^{\hat{g}} = \tilde{\alpha}^{\hat{g}} - \alpha^f (b\hat{c} + \hat{c}\rho)\right\}.$$

The stability and determinacy properties of rules associated to elements of Ω can be characterized numerically. As an example, consider Figure 3. Here the Woodford calibration is used together with the Medium Lag inertial specification, $\beta = .98$, $\psi = 10$ and the timeless perspective is assumed. To create the figure, a $(-1, 2) \times (-1, 2)$ lattice was imposed over the $(\alpha^f_\pi, \alpha^f_x)$-space, and at each point on the lattice, the associated element of Ω was determined and the stability and determinacy properties were recorded. The center of the large gray dot indicates the location of the EH-rule, which lies in the interior of the stable determinate region.

Figure 3 Here

We have chosen this admittedly extreme case to emphasize that while the EH-rule necessarily lies in the region of stable determinacy, it may be very near a boundary and surrounded by all types of bad outcomes: unstable determinacy and stable and unstable indeterminacy are all possibilities. We conducted similar analysis on a $(-5,5)\times(-5,5)$ lattice in $(\alpha_\pi^f,\alpha_x^f)$-space for all combinations of calibration, inertia specification, β-value, ψ-value, and FOC-type, and we found the results of Figure 3 to be qualitatively universal, with the caveat that the MN-calibration with various permutations of the remaining parameters may yield relatively large regions of stable determinacy, and may house the EH-rule far from any problem region.

Analysis of the policy manifold indicates that arbitrarily choosing an optimal interest-rate rule among those available in Ω is unwise, for instability or indeterminacy may result. These results also suggest that relying on an unconstrained numerical algorithm to search for the optimal rule is ill-advised for such an algorithm will be unable to distinguish between points on the manifold – all points yield the same value of the government's objective. These searches must be constrained to those regions in policy space corresponding to stable determinacy.[14]

4.4 Model Uncertainty

While analysis of the policy manifold indicates potential problems for optimal interest-rate rules, one may wonder about the relevance of these concerns given the existence of the EH-rule, which yields stable determinacy for all calibrations. And indeed if the structural parameters of the AS and IS relations are known with precision, the EH-rule can be implemented and no concern over potential indeterminacy or instability problems is warranted. On the other hand, model uncertainty, which here takes the form of uncertainty about the true values of the model's structural parameters, may imply bad outcomes even when the EH-rule is employed. To argue this point, notice that the set of all policy rules (17) may be identified with $\mathbb{R}^6$. The optimal policy manifold is precisely the subset of $\mathbb{R}^6$ coinciding with those rules which implement the relevant FOC. A simple "continuity of eigenvalues" argument shows that since the optimal policy manifold has subsets corresponding to instability, indeterminacy, and stable sunspots, which are non-empty and open in the *relative* topology, there must be corresponding non-empty open subsets of $\mathbb{R}^6$ whose associated policies induce indeterminacy, instability or stable sunspots. Now suppose a policy maker chooses to use the EH-rule by estimating the structural parameters of the model and setting policy accord-

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[14] This point was emphasized for other interest-rate rules in Evans and McGough (2007).

ingly. The policy maker thinks she is choosing a point on the optimal policy manifold and in a region corresponding to stable determinacy. However, if her estimates of the structural parameters are off, it is very likely she is not on the manifold, and, much more importantly, because the location of the open sets in $\mathbb{R}^6$ corresponding to stable determinacy are not where she thinks they are, the associated model may be unstable, indeterminate, or have stable sunspot equilibria. As a concrete example of this phenomenon, we note that if the policy maker with $\psi = .1$ thinks the MN calibration with medium lag prevails and sets policy according to the associated EH-rule under the timeless perspective, but if in fact the true calibration is W or CGG then the model may exhibit either unstable indeterminacy, stable indeterminacy, or explosiveness depending on the inertial specification.

These problems indicate that, in the case of model uncertainty, the optimal rule should be chosen to have nice stability and determinacy properties across possible model specifications, as well as to maximize some measure of welfare. A technique for determining these types of rules is provided in Evans and McGough (2007).

5 Conclusion

In our view, the analysis of optimal monetary policy should include a stand on implementation. In this paper we take a stand by requiring our policy makers to commit to an instrument rule. This specification of policy-maker behavior allows us to characterize the collection of all instrument rules consistent with the implementation of optimal policy; and further, it allows us to rank-order the rules according to the number and nature of the associated equilibria. We find that while rules imparting determinacy and stability, that is, generalized EH-rules, always exist, they may lie very near policy rules that result in instability, indeterminacy, or even stable sunspots. These findings suggest that in the presence of structural parameter estimation error, bad outcomes may obtain even in case the EH-rule is employed. This serves as a strong caution to policymakers and suggests policy that is robust to parameter uncertainty and other types of model uncertainty is important.

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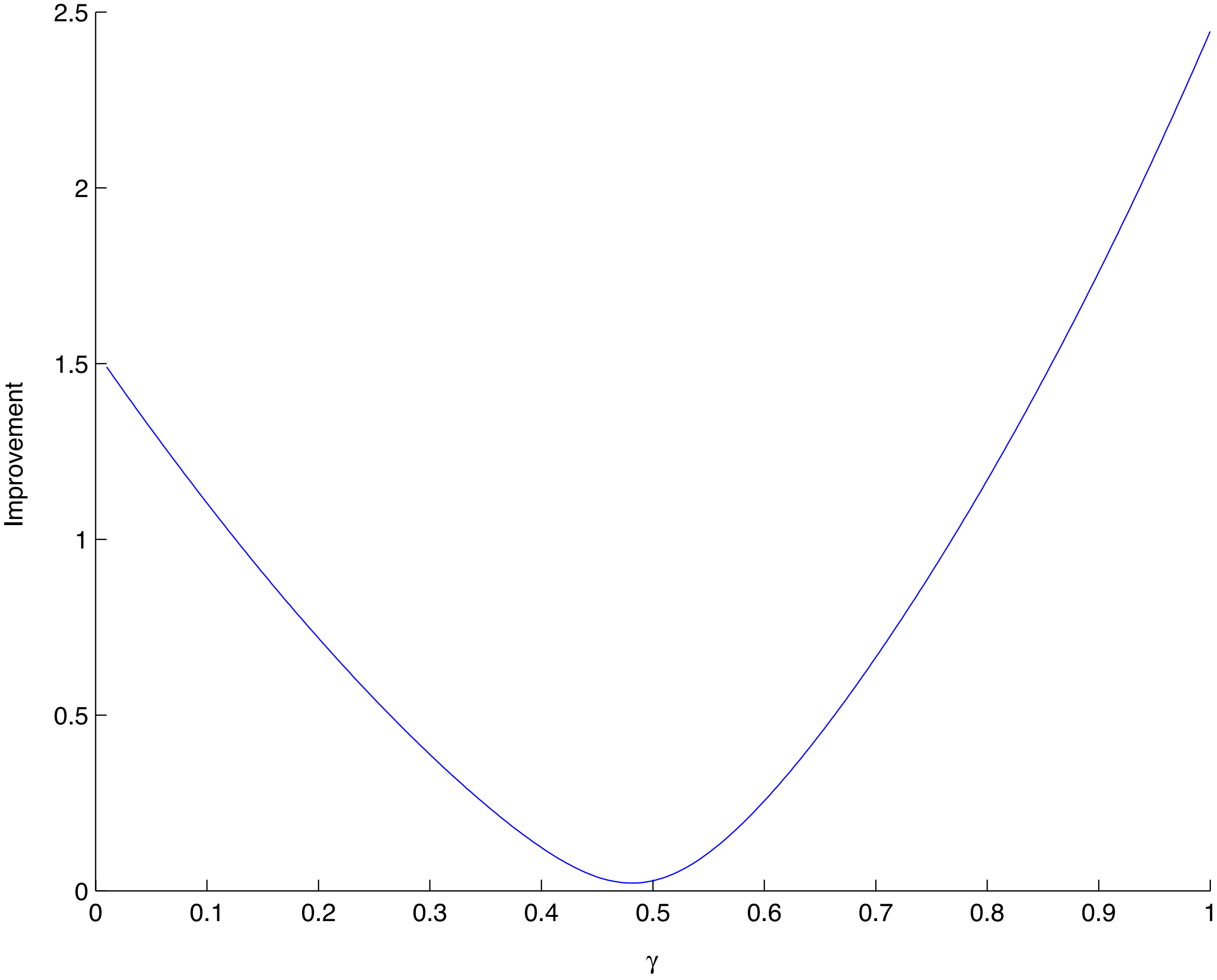

Figure 1: MJB/TP Improvement for Varyingγ
Woodford calibration, ψ=1

Figure 2: MJB/TP Improvement:
Woodford calibration, ψ=1

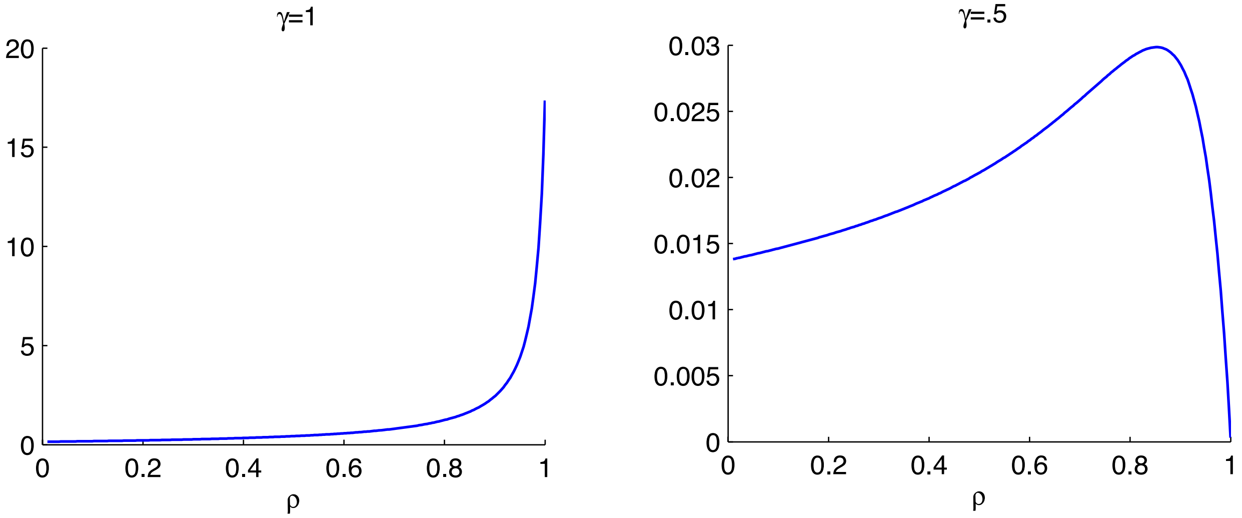

Figure 3: Policy Manifold, Timeless Perspective

ϕ=6.37, λ=.024, γ=.5, δ=.5, β=.98, ψ=10