Consistent Expectations and Misspecification in Stochastic Non-linear Economies*

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Abstract

This paper generalizes existence results on first-order Stochastic Consistent Expectations Equilibria (SCEE) obtained by (Hommes, Sorger, and Wagener 2002). We present a stochastic non-linear self-referential model in which expectations are based on linear perceptions. In an SCEE the correlation coefficients of the true and perceived processes coincide. We provide conditions on the non-linear maps that govern the stochastic process sufficient to establish existence of SCEE. Our approach defines a map that takes linear perceptions to actual outcomes. We show that (i.) a non-trivial fixed point to this map exists, and, (ii.) all fixed points are SCEE. We provide conditions under which these equilibria are E-stable. Simulations show that E-stability governs real-time learning.

1 Introduction

One criticism of the rational expectations hypothesis is that its assumption requires agents know the economy's underlying distribution. Recently, the rational expectations hypothesis has been replaced with equilibrium models incorporating misspecification (e.g.,(Evans, Honkapohja, and Sargent 1993), (Evans and Honkapohja 2001), (Sargent 1999), and (Branch and Evans 2002)).

An interesting and new equilibrium concept called a *Stochastic Consistent Expectations Equilibrium* (SCEE) has been developed by (Hommes and Sorger 1998) and (Hommes, Sorger, and Wagener 2002). In an SCEE the true process is defined by a (unknown) non-linear self-referential map. Agents make forecasts via a linear

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perceived law of motion (PLM). An SCEE obtains when the correlation coefficients of the non-linear stochastic process coincide with those predicted by the PLM.

Motivation for the notion of an SCEE comes from a desire to instill agents with feasible, as well as in some sense optimal, forecasting mechanisms. Many economic models such as overlapping generations models (OLG), asset pricing models, and stochastic general equilibrium models of the business cycle follow non-linear laws of motion. In these models it is very difficult even for specialists to obtain representations of the associated rational expectations equilibria which are useful for making optimal forecasts, without first linearizing the model. The idea put forth by (Hommes and Sorger 1998) and (Hommes, Sorger, and Wagener 2002) is to simply assume that agents make forecasts using linear models, and then ask whether it is possible for these linear models to be consistent with the data produced by the underlying non-linear model. When such consistency obtains, the agents will be behaving optimally in the sense that their linear model is the best among all linear models.

In an SCEE agents make systematic forecast errors. However, these errors are not serially correlated. It is important to note then that agents' forecast errors– coming from mistakenly assuming a linear structure– are uncorrelated with their linear beliefs. In this sense agents are unable to detect their misspecification as simple econometric tests would reinforce their linear model. An SCEE, therefore, also satisfies one of the primary criteria cited in favor of rational expectations; namely, that forecast errors are serially uncorrelated. Indeed serial uncorrelation of forecast errors is a standard test of the rational expectations hypothesis.

Our approach closely follows (Hommes, Sorger, and Wagener 2002). In that paper they specify dynamics that follow a non-linear self-referential map

$$x_t = F(x^e_{t+1}, \eta_t), \tag{1}$$

where x^e_{t+1} are expectations of x_{t+1} formed at time t, and η_t is an iid zero-mean process. (Hommes, Sorger, and Wagener 2002) depart from the rational expectations hypothesis and instead suppose beliefs are formed based on the linear law of motion

$$x_t = c + bx_{t-1} + \epsilon_t.$$

An SCEE is a stationary process which solves (1), $E(x_t) = \frac{c}{1-b}$, and $b^j = corr(x_t, x_{t-j})$.

(Hommes, Sorger, and Wagener 2002) consider the existence of SCEE. They show that if F is bi-linear with slope-zero then an SCEE exists with non-trivial b.[1] This functional form does not fit the OLG model of (Grandmont 1985), the desired laboratory. So (Hommes, Sorger, and Wagener 2002) present numerical evidence that SCEE exist in an OLG model. In this paper we generalize the existence argument of (Hommes, Sorger, and Wagener 2002). We provide conditions on the map F sufficient to guarantee existence of an SCEE.

We further show that if agents have linear beliefs that do not depend on a constant c the obtained sufficient conditions will guarantee existence of an SCEE with a

[1] That is, with non-zero autocorrelation.

non-trivial (i.e. non-zero) b. Moreover, these SCEE will be E-stable, implying that reasonable learning schemes will converge to an SCEE. We show, somewhat more weakly, that when beliefs depend on a constant c, the sufficient conditions guarantee an SCEE exists, though it may only exist in uncorrelated processes (i.e. $b = 0$). We also present numerical evidence that when $c \neq 0$ SCEE with $b = 0$ may be the only stable equilibria. We engage in a discussion which argues that the obtained sufficient conditions are not necessary for existence, and that alternative specifications, not covered by our analytical results, may yield non-trivial SCEE.

While the model we consider (a variant of (Hommes, Sorger, and Wagener 2002)) is fairly general, the conditions that we require F to satisfy for existence of non-trivial SCEE to be guaranteed are admittedly not satisfied by standard specifications of, for example, the OLG-model or the asset pricing model mentioned above. However, we hope these conditions will facilitate applications to other, more specific economic models. Indeed, this paper provides a general approach for establishing existence of SCEE in many economic models, such as OLG, asset pricing, real business cycle models, New-Keynesian general equilibrium models, among others. We are optimistic that the methods established here can be extended to incorporate these other functional specifications.

It should be noted that Stochastic Consistent Expectations Equilibria fit into a larger class of equilibria called Restricted Perceptions Equilibria (RPE). RPE were first considered by (Evans, Honkapohja, and Sargent 1993) and generalized in (Evans and Honkapohja 2001). In an RPE, agents misspecify, in some dimension, but form their beliefs optimally given this misspecification. For example, in (Evans and Honkapohja 2001) and (Branch and Evans 2002) agents underparameterize their perceived law of motion. In (Evans, Honkapohja, and Sargent 1993) a fraction of agents mistakenly assume that the true non-linear model is actually linear.[2] Optimality of these misspecified beliefs is imposed via an orthogonality condition. An SCEE is similar to an RPE in that all agents mistakenly believe the non-linear model is linear. Expectations are consistent given this misspecification since the correlation coefficients of their perceived model exactly coincide with the non-linear model.

This paper proceeds as follows. In Section 2 we present the existence argument for the case where beliefs do not have a constant. Section 3 presents the argument extended to the case where beliefs do depend on a constant, though we are not able to conclude analytically that non-trivial SCEE exist. This statement will be made more precise in Section 3. Section 4 concludes.

[2]The model in (Evans, Honkapohja, and Sargent 1993) is based on the same OLG model of (Grandmont 1985) that is considered in (Hommes, Sorger, and Wagener 2002). In Evans, Honkapohja, and Sargent, however, the interest is whether cycles can be preserved if a fraction of agents believe the model is linear.

2 Existence and Stability of SCEE

In this section we use a T-map – a map that takes perceptions to implications – to show the existence of order one *stochastic consistent expectations equilibria* in a simple non-linear forward looking model. The model is given by

$$x_t = F(x^e_{t+1}) + \eta_t \tag{2}$$

where x_t is univariate, and η_t is zero mean *i.i.d.* with full support, taking on values in $[-\overline{\eta}, \overline{\eta}]$. Models of this form are prevalent in dynamic macroeconomics: for example, (Grandmont 1985), (Guesnerie and Woodford 1991) consider overlapping generations models of the form (2).

We focus here on the case that F is symmetric about the origin and that agents believe the constant term in their linear perceptions to be zero. We relax this assumption in Section 3, allowing agents to have perceptions that include a constant term, and, as a consequence, we obtain weaker results.

The notation x^e_{t+1} captures agents expectations of x_{t+1} formed at time $t-1$. We impose that agents form these expectations using the perceived law of motion

$$x_t = bx_{t-1} + \varepsilon_t \tag{3}$$

and so that ε_t is not assumed known. Thus $x^e_{t+1} = b^2 x_{t-1}$. The resulting dynamic system, called the actual law of motion, is given by

$$x_t = F(b^2 x_{t-1}) + \eta_t. \tag{4}$$

(Hommes, Sorger, and Wagener 2002) pose the following question: If agents have linear beliefs (3) and if the state variable follows the non-linear reduced form model (2), does there exist a belief parameter b that is linearly consistent with the associated stochastic process x_t? The following is a precise definition of the equilibrium.

Definition 1 *The pair $(\{x_t\}, b)$ is a (first order) stochastic consistent expectations equilibrium (SCEE) if $E(x_t) = 0$ and $b = corr(x_t, x_{t-1})$.*

We require the following restrictions on F:

A.1 F is twice continuously differentiable with $F' > 0$ and $sgn(F''(x)) = -sgn(x)$.

A.2 F is symmetric about the origin.

A.3 If $F'(0) > 1$ then there exists $x^* > 0$ so that $x > x^* \Rightarrow F(x) < x$.

A.4 If $F'(0) > 1$ then $\overline{\eta} > \sup_{0 \le x \le x^*} F(x) - x$.

A.5 If $F'(0) > 1$ then $a' = \inf\{x > x^* | x - F(x) > \overline{\eta}\}$ exists. If $F'(0) < 1$ then $a' = \inf\{x > 0 | x - F(x) > \overline{\eta}\}$ exists.

The primary contribution of this paper is to provide more general analytical results on the existence of SCEE. Our approach is sufficiently general so to be useful in applications to a variety of economic models. Our proof of existence proceeds as follows: we begin by showing that for any $b \in [0, 1]$, the process (4) is asymptotically stationary.[3] Then, given b, we define $T(b) = E(x_t^2)^{-1}E(x_t x_{t-1})$, noting existence of these expectations is guaranteed by stationarity. The next step is to show T is continuous and takes the interval $[0, 1]$ into itself. That a fixed point exists is then guaranteed by Brouwer's theorem, but, in fact, it is clear that $b = 0$ is a fixed point, and so more work is required. We show non-trivial fixed points of the T-map exist given a further restriction on the function F, and finally that any given fixed point is an SCEE is shown proving $E(x_t) = 0$; a result that follows from the symmetry of F.

2.1 Asymptotic Stationarity

Let $a = a' + \overline{\eta}$ and notice that $b \in [0, 1]$, $x_{t-1} \in [-a, a]$ implies $x_t \in [-a, a]$.[4]

To prove stationarity results we rely on the theory developed in Stokey and Lucas. Our notation is slightly different from theirs because of conflicts. The transition function P_b associated to the dynamic stochastic system (4) is defined by

$$P_b(x, A) = prob\{x_{t+1} \in A | x_t = x\}.$$

We may then define the two operators S and S^* (denoted T and T^* by Stokey and Lucas) as follows: for bounded measurable functions f, we have

$$Sf(z) = \int f(x) P_b(z, dx);$$

and for measures λ we have

$$S^*\lambda(A) = \int P_b(x, A) d\lambda(x).$$

The measure $S^*\lambda$ gives the distribution of x_{t+1} given that x_t has distribution λ. Notice that if $S^*\lambda = \lambda$, and for all initial distributions λ_0 we have $S^{*n}\lambda_0 \to \lambda$ weakly, then x_t is asymptotically stationary; thus we study the behavior of $S^*\lambda$.

A transition function is said to have the Feller property provided the associated operator S takes the set of bounded continuous functions to itself. Because F is continuous, it follows that our transition functions P_b have the Feller property. (Stokey and Lucas, p.237)

[3] This proof relies on the monotonicity of the transitions functions, which, in turn, relies on $F' > 0$. To our surprise, we are unable to find analogous theorems guaranteeing asymptotic stationarity in case $F' < 0$, though our intuition suggests it holds.

[4] Consider the case $x_{t-1} \geq 0$. If $x_{t-1} \leq x^*$ then $x_t \leq x^* + \overline{\eta} \leq a' + \overline{\eta}$; if $x_{t-1} \in [x^*, a']$ then $x_t \leq x_{t-1} + \overline{\eta} \leq a' + \overline{\eta}$ and finally if $x_{t-1} \geq a'$ then $x_t \leq x_{t-1}$.

A transition function is said to be monotone if whenever f is bounded and increasing, so too is Sf. Stokey and Lucas (exercises 12.9 and 12.11) show that P_b is monotone provided

$$P_b(s, (-\infty, x]) \leq P_b(s', (-\infty, x])$$

whenever $s > s'$. In the Appendix we use this sufficient condition to show P_b is monotone.

Lemma 2 *The transition function P_b is monotone.*

The transition function P_b is said to satisfy the mixing axiom provided there is a $d \in [-a, a]$, $\varepsilon > 0$ and $N \geq 1$ so that

$$P_b^N(-a, [d, a]) > \varepsilon \quad \text{and} \quad P_b^N(a, [-a, d]) > \varepsilon.$$

In the Appendix, we show that property 4 of F guarantees the mixing axiom is satisfied.

Lemma 3 *The transition function P_b satisfies the mixing axiom.*

Proposition 4 *The process x_t defined by (4) is asymptotically stationary.*

This follows from Lemmas (2), (3), and that P_b satisfies the Feller property. See Theorem 12.12 in Stokey and Lucas.

2.2 Continuity

Above it was shown that for each $b \in [0, 1]$, the process x_t defined by (4) converges weakly to a unique asymptotic distribution, which we now denote λ_b. Now let $s_n \to s$ in $[-a, a]$ and $b_n \to b$.

Lemma 5 *$P_{b_n}(s_n, \cdot) \to P_b(s, \cdot)$ weakly.*

For proof, see Appendix.

Proposition 6 *If $b_n \to b$ then $\lambda_{b_n} \to \lambda_b$ weakly.*

This proposition follows directly from Lemma 5 and Theorem 12.13 in Stokey and Lucas.

We now define the map $T : [0, 1] \to \mathbb{R}$ as follows:

$$T(b) = E_{\lambda_b}(x_t^2)^{-1} E_{\lambda_b}(x_t x_{t-1}) \tag{5}$$

where $E_{\lambda_b}(f(x_t))$ is the expectation of f taken with respect to the limiting distribution of the process x_t as defined by (4), namely λ_b.

Proposition 7 *The map T is continuous.*

This proposition follows immediately from Proposition 6 and the fact that $E_{\lambda_b}(x_t x_{t-1}) = E_{\lambda_b}(x_{t-1} F(b^2 x_{t-1}))$.

2.3 Existence of SCEE

To prove existence, we first show that T has a fixed point.

Lemma 8

1. *If $b \geq 0$ then $T(b) \geq 0$.*
2. *If $b \leq 1$ then $T(b) \leq 1$.*

For proof, see Appendix. It follows immediately from the above Lemma and Brouwer's fixed point theorem that T has a fixed point in $[0, 1]$.

Proposition 9 *A fixed point to the T map is an SCEE.*

For proof, see Appendix.

Proposition 9 shows that it is possible for agents' linear beliefs to be consistent with a non-linear process. Examination of (4) makes it clear why a fixed point to T is an equilibrium. Agents have linear beliefs parameterized by b. These beliefs, in turn, affect the stochastic process x_t and its correlation coefficient. An equilibrium occurs when these beliefs lead to a process whose correlation coefficient reinforces those beliefs. The sufficient conditions provided guarantee that such an equilibrium exists.

The above proposition guarantees the existence of an SCEE. But this existence can be established much more easily; the pair $(\eta_t, 0)$ is an SCEE, and, of course, zero is a fixed point of the T-map. The question remains, "Do there exist non-trivial fixed points?" The following results address this question.

Proposition 10 *If $F' < 1$ then the only fixed point of the T-map is zero.*

For proof, see Appendix.

Proposition 11 *If there exists $\hat{x} > 0$ so that $a < F(\hat{x})^2/\hat{x}$ then there exists $b > 0$ so that $T(b) = b$.*

For proof, see Appendix.

Note that the condition $a < F(\hat{x})^2/\hat{x}$ is somewhat complicated because the definition of a depends on the support of η. Intuitively, it requires that, as x goes to zero, the average value of F be increasing faster than the value of F is decreasing. The existence of F satisfying this condition is easy to establish: let $F(x) = x^{1/3}$. Also, it is shown in the Appendix that the existence of $\hat{x}$ guarentees $F'(0) > 1$, thus distinguishing this proposition from the previous one.

We again argue that an SCEE is a reasonable equilibrium concept. Most economic models are non-linear or linearized versions of non-linear models. One might ask when a linearization is a 'good' approximation to a non-linear model. Econometricians often say that a linear specification is appropriate when non-linearity is undetectable using linear statistical tests. An SCEE extends this criterion to economic agents; see (Hommes, Sorger, and Wagener 2002) for more details.

2.4 E-Stability

We conjecture that the E-stability principle holds for these models; that is, if agents postulate linear beliefs and update these beliefs in a reasonable manner, then the agents will learn an SCEE provided the SCEE is E-stable. E-stability has been shown to be an important criteria for equilibrium selection in many models.[5] Because we do not know if our T map is differentiable we modify the usual definition of E-stability slightly. A fixed point b is E-stable provided there is a neighborhood of b so that in that neighborhood, whenever $b' > b$ it follows that $T(b') < b'$ and whenever $b' < b$ it follows that $T(b') > b'$. Intuitively, the T map must move the parameters in the direction of the fixed point. It may occur that the set of fixed points has non-trivial connected components. Recall that a connected subset of $\mathbb{R}$ is an interval. Suppose I is an interval of fixed points. Consistent with Evans and Honkapohja, we say the interval I is E-stable if whenever $b' < \inf I$ it follows that $T(b') > b'$ and whenever $b' > \sup I$ it follows that $T(b') < b'$.

Proposition 12 *If $F' < 1$ then the fixed point $b = 0$ is E-stable.*

This follows immediately from the proof of proposition 10 above.

Proposition 13 *If there exists $\hat{x} > 0$ so that $a < F(\hat{x})^2/\hat{x}$ then there is a non-trivial E-stable fixed point or set of fixed points.*

This follows immediately from the proof of proposition 2.3 above and the continuity of T. We remark that $b = 0$ is still an SCEE when there exists a non-trivial

[5] See (Evans and Honkapohja 2001) for an extensive analysis.

fixed point. Moreover, we can not rule out that $b = 0$ is E-stable. This raises the interesting possibility of multiple E-stable fixed points.

These results suggest than an SCEE will be stable under an appropriate learning algorithm. The next subsection will present two examples. We remark that E-stability is a robustness check of an SCEE. Even if agent' initial perceptions are out of equilibrium they will still learn the true correlation coefficients. Thus, an SCEE is not just an equilibrium anomaly, but in these models one can reasonably expect agents to have consistent expectations given their misspecification.

2.5 Real Time Learning

In this section we analyze the behavior of the economy under real time learning. Specifically, agents estimate the coefficients of the linear PLM and use these estimates to form expectations. These expectations generate new data via the reduced form model, and agents use these new data to update their estimates. As usual, the relevant learning question is, "Do agents' estimates converge to a fixed point of the T map?"

Analysis of the asymptotic behavior of recursive estimators typically depends on the theory of stochastic approximation; and there is a subset of the theory – that part dealing with non-conditionally linear Markovian state dynamics – that applies to our model. Unfortunately, little is known about the functional form of the T-map, and thus the relevant regularity conditions required for application of the theory can not be verified.

Given the intractability of analytic results, we turn to simulations. We begin by specifying a functional form for F and then impose that agents update their estimates one of two ways: either by OLS (RTL Method 1) or by sample estimation of the correlation coefficient (RTL Method 2), as suggested by (Hommes, Sorger, and Wagener 2002). We choose $F(x) = x^{1/3}$ and $\overline{\eta} = 1.1$. OLS estimates are updated according to the algorithm

$$\begin{aligned} b_t &= b_{t-1} + t^{-1} R_t^{-1} x_{t-1} \left(x_t - b_{t-1} x_{t-1} \right) \\ R_t &= R_{t-1} + t^{-1} \left(x_t^2 - R_{t-1} \right) \end{aligned}$$

where R_t is the sample second moment of x_t. Sample estimation of the correlation coefficients follows

$$b_t = \frac{\sum_{T=1}^{t-1} x_T x_{T-1}}{\sum_{T=1}^{t} x_T^2}.$$

This particular updating algorithm was suggested by (Hommes, Sorger, and Wagener 2002) in order to guarantee $b \in [0, 1]$.

We compute the fixed point of the $T - map$ by running 2000 independent simulations of 1000 periods each. The final two realizations of each simulation were used to compute the relevant variance, and covariance, and hence determine the correlation coefficient.

We also ran repeated simulations to determine the asymptotic behavior of the economy under the two different learning schemes. We randomly draw initial values for b and x and simulate the model for 5000 periods. Figure 1 plots a typical simulation of real-time learning under RTL Method 1. The horizontal line in the plot is the numerically computed non-trivial fixed point of the T-map. As can be seen from the plot, agents perceptions appears to converge to the fixed point. Figure 2 plots a typical simulation of real-time learning under RTL Method 2. It also appears to converge to the fixed point. Using either method of learning, we find that a non-trivial fixed point appears to be stable. Specifically, the estimators appear to converge to $b \approx .92$, and, numerically, this value of b appears to be a fixed point of T.

INSERT FIGURE 1 HERE

INSERT FIGURE 2 HERE

3 SCEE with a Constant Term

In the previous section we directed attention to reduced form functions that are symmetric about the origin. This restriction allowed us to impose perceived laws of motion which included no constant term. It is, however, more natural to relax this restriction and allow agents to regress on a constant term as well. In this section we begin by showing the technique of defining a T-map extends to this more general case, thus giving analytic support to the numerical method of computing fixed points. Unfortunately, we will find that we are unable to show existence of SCEE with non-zero autocorrelation and, further, present numerical evidence suggesting that under the restrictions imposed on the function F, SCEE with non-zero autocorrelation may not exist. We will then discuss alternative restrictions on F that may allow existence, and again present supporting numerical evidence.

3.1 The T-map Revisited

Again we consider the model given by

$$x_t = F(x^e_{t+1}) + \eta_t \tag{6}$$

where x_t is univariate, and η_t is zero mean *i.i.d.* taking on values in $[-\overline{\eta}, \overline{\eta}]$.

Now, we impose that agents form their expectations using the perceived law of motion

$$x_t = c + bx_{t-1} + \varepsilon_t \tag{7}$$

and so that ε_t is not assumed known. Thus $x^e_{t+1} = c(1+b) + b^2 x_{t-1}$. The resulting dynamic system, called the actual law of motion, is given by

$$x_t = F(c(1+b) + b^2 x_{t-1}) + \eta_t. \tag{8}$$

Again, our charge is to determine under what conditions, if any, there exist values c and b that are linearly consistent with the ALM. We now alter our definition of an SCEE to this more general case.

Definition 14 *The triple $(\{x_t\}, c, b)$ is a (first order) stochastic consistent expectations equilibrium (SCEE) if x_t is asymptotically stationary and generated by the recursion (8), $E(x_t) = c/(1-b)$, and $corr(x_t, x_{t-1}) = b$.*

To analyze existence of such equilibria, we proceed precisely as we did in the previous section.

The restrictions on F are similar to the above:

A.1 F is twice continuously differentiable with $F' > 0$.

A.2 F is symmetric about α.

A.3 F has horizontal asymptotes $\pm h$.

A.4 $sgn(F''(x-\alpha)) = -sgn(x-\alpha)$.

A.5 Let $G(x) = F(x+\alpha)$. If $G'(0) > 1$ let $x^* > 0$ so that $x > x^* \Rightarrow G(x) < x$ (existence is guaranteed by horizontal asymptotes,) and assume $\overline{\eta} > \sup_{0 \le x \le x^*} G(x) - x$.

To examine existence, we define a $T-map$, much like before, and, as before, this requires asymptotic stationarity.

3.1.1 Asymptotic Stationarity and Continuity

Fix $c \in \mathbb{R}$ and $b \in [-1, 1]$ respectively; we must allow negative values of b for reasons discussed below. We employ the same notation as above, except the transition function is now labeled $P_{c,b}$. Because of the horizontal asymptotes, we may set $a = h + \overline{\eta}$ and conclude $x_t \in [-a, a]$. That this transition function satisfies the Feller property and monotonicity follows from continuity and from $F' > 0$ respectively, just as before.[6] Showing the mixing axiom holds is more challenging.

Lemma 15 *The transition function $P_{c,b}$ satisfies the mixing axiom.*

[6]Negative values of b do not effect the proofs because the coefficient on x is always b^2.

See the Appendix for a proof.

We may conclude, as before, with the following result:

Proposition 16 *The process x_t defined by (8) is asymptotically stationary.*

Denote by $\lambda_{c,b}$ the stationary distribution associated to the process x_t for given c and b. Define $T : [-2a, 2a] \times [-1, 1] \to \mathbb{R}^2$ by $T(c, b) = (T_a(c, b), T_b(c, b))$, where[7]

$$\begin{aligned} \mu_{\lambda_{c,b}} &= E_{\lambda_{c,b}} x_t \\ T_b(c,b) &= (E_{\lambda_{c,b}}(x_t - \mu_{\lambda_{c,b}})^2)^{-1} E_{\lambda_{c,b}}(x_t - \mu_{\lambda_{c,b}})(x_{t-1} - \mu_{\lambda_{c,b}}) \\ T_c(c,b) &= \mu_{\lambda_{c,b}}(1 - T_b(c,b)) \end{aligned}$$

This map is well-defined by Proposition 16.

Recall that continuity of the T-map in the previous section effectively followed from the continuity of the function F. Continuity of the T-map in this section is entirely analogous. Let $(c_n, b_n) \to (c, b)$. An argument almost identical to the ones given in Lemma 5 and Proposition 6 shows that λ_{c_n,b_n} converges weakly to $\lambda_{c,b}$. We may conclude, as before, with the following result.

Proposition 17 *The map T is continuous.*

3.1.2 Existence of SCEE

By construction, a fixed point of the T-map is a stochastic consistent expectations equilibrium. Since T is continuous, to show that it has a fixed point, we must simply show that it maps the compact set $[-2a, 2a] \times [-1, 1]$ into itself.

This is established in the following lemma.

Lemma 18 $T([-2a, 2a] \times [-1, 1]) \subset [-2a, 2a] \times [-1, 1]$

For a proof, see the Appendix.

Apply the Brouwer theorem to conclude that a fixed point of the T-map, and hence an SCEE, exists.[8]

[7] Of course, below, we will show the range of T is in $[-2a, 2a] \times [-1, 1]$.

[8] Intuition suggests that since $F' > 0$, it should follow that $T_b \geq 0$ whenever b is. Unfortunately, the proof alludes us; therefore we can not analytically rule out SCEE with negative correlation coefficients.

3.2 Existence of Non-Trivial SCEE?

As before, it is not necessary to appeal to Brouwer's theorem to obtain existence of trivial SCEE; that is, SCEE for which $b = 0$. In particular, let $c = F(c)$; then $T(c, 0) = (c, 0)$. In the previous section we were able to place further restrictions on F and then appeal to the Brouwer theorem to obtain a non-trivial fixed point. A similar argument for the more general case analyzed in this section has proved elusive.

It is natural, then, to turn to numerical methods to estimate fixed points of the T-map. Unfortunately, computing specific values of T using sample means and autocorrelations, and the subsequent search for a fixed point has proven too costly: the number of simulations required to get accurate estimates is prohibitive when searching is required. In the previous section, we found that a much more efficient way of computing an SCEE was to have agents attempt to learn the SCEE values through recursive estimation. We find that a similar experiment for the more general model is enlightening.

We choose the same functional form of F as before, that is $F(x) = (x - \alpha)^{1/3}$, except that we artificially impose asymptotes for x of large magnitude. Agents are given the PLM (7) and are assumed to use recursive least squares to form their estimates. Figure 3 exhibits a sample simulation with $\alpha = .5$. The final value of b is very near zero, and the final value of c is 1.39, which is approximately a fixed point of the function F. From this simulation and others using different initial conditions, different values of the shift parameter α, and even different functional forms, we conclude that even if non-trivial SCEE exist, they do not appear to be stable under learning. It is interesting to note that even when $\alpha = 0$, so that the corresponding function is symmetric about the origin, the only stable SCEE is trivial. Of course if agents perceive a non-zero constant term a, the actual law of motion will be governed by a non-linear function that is not symmetric about the origin.

INSERT FIGURE 3 HERE

In fact, some reflection suggests non-trivial SCEE may not even exist when the function F is increasing and not symmetric about the origin. To gain some intuition, first notice that the restrictions imposed on F in this section are not necessary for the associated process to be asymptotically stationary; they were imposed for expositional purposes as well as to guarantee that the associated T-map took a compact set to itself. For example, consider the linear map $F(x) = mx + h$, with $0 < m < 1$. For given perceptions (c, b), the resulting process is given by

$$x_t = m(1 + b)c + h + mb^2 x_{t-1} + \eta_t.$$

Obviously, if $|b| \leq 1$, this process is asymptotically stationary, and furthermore, $T_b(c, b) = b^2 m$. But this implies that $T_b(c, b) = b$ if and only if $b = 0$, and thus the only SCEE in this case is trivial. Finally, notice that $b > 0$ implies $T_b(c, b) = b^2 m < b$

so that the implied value of b is actually driven to zero under learning. This explains why the trivial SCEE appear to be stable.

Similar intuition holds if the function F is increasing and non-linear. Let $F_{c,b}(x) = F(c(1+b)+b^2x)$. If $F_{c,b}$ has only one stable fixed point, and if $\overline{\eta}$ is not too large, the asymptotic behavior of x_t will be much like that determine by the tangent of $F_{c,b}$ at the fixed point: this will tend to drive the implied autocovariance to zero, just as occured above when F was assumed linear.

The idea is that the constant term shifts the non-linear function F left. As it does so, there is only one stable fixed point to F and, with appropriate restrictions on the support of η, a linear-approximation is a good representation of the dynamics. However, when there is no constant, so that F and the ALM are symmetric about the origin, there are multiple stable fixed points. Thus, it is possible to have non-linearity in the T-map and non-trivial SCEE.

Our inability to confirm the existence of non-trivial SCEE seems to stem from two sources: 1) the assumption that F is increasing, and 2) the fact that if an agent believes there is no autocorrelation, then there will be no autocorrelation. In fact, (Hommes, Sorger, and Wagener 2002) have reported strong numerical evidence for the existence of stable non-trivial SCEE in case the function F is a hump shaped non-linear map as derived from Grandmont's OLG model. This suggests that simply eliminating the requirement that F be increasing may lead to non-trivial SCEE; unfortunately, the assumption that F is increasing is required for our proof that the transition functions of the stochastic process satisfy the monotonicity property, which, in turn, leads to asymptotic stationarity. The assumption of F increasing is important to the analytical results but it is also a barrier to the existence of multiple stable fixed points.

To investigate whether altering point 2 above will have an effect on existence, we proceed as follows: Specify a new reduced form model

$$x_t = F(x^e_{t+1}) + \gamma x_{t-1} + \eta_t.$$

If agents have the PLM

$$x_t = c + bx_{t-1}$$

then the actual law of motion will be given by

$$x_t = F(c(1+b) + b^2x_{t-1}) + \gamma x_{t-1} + \eta_t.$$

In this case, $b = 0$ does not imply zero autocorrelation because of the term γx_{t-1}. We simulated this model using the same functional form for F as above, and with $\alpha = -1$ and $\gamma = .5$; a sample run is presented in Figure 4. We see that the agents' perceptions appear to converge to $(1.5, .54)$. And, indeed, if we plot sample data, together with the regression line, we see that the fit looks remarkably good: see Figure 5. On the other hand, this result is not surprising given the form of the function F. For perceptions (c, b), the associated $F_{c,b}$ is

$$F_{c,b}(x) = b^{2/3}(x + c(1+b)/b^2 - \alpha)^{1/3},$$

which, given our numbers, is the function $x^{1/3}$ shifted left by 8.9 and scaled down by .66. Thus F is very nearly linear near its fixed point; in fact, if F is plotted in same region as represented in Figure 5, the graph of F and the graph of the regression line nearly coincide. We find it interesting that, in this case, the linear beliefs of agents forces a model, which is in general non-linear, to behave as if it were linear.

INSERT FIGURES 4 & 5 HERE

4 Conclusion

This paper considered the existence of Stochastic Consistent Expectations Equilibria (SCEE) for certain stochastic processes that depend on increasing non-linear laws of motion, and provides sufficient conditions for existence of such equilibria. We present a technique which generalizes the existence argument of (Hommes, Sorger, and Wagener 2002) to a more general class of functions.

When agents beliefs are based on a linear law of motion but the true process depends on a non-linear map, an equilibria exists (provided certain regularity conditions are satisfied) in which the correlation coefficients of the perceived model are identical to the asymptotic correlation coefficients of the true stochastic process. We show that when agents' linear beliefs do not depend on a constant, an SCEE exists with non-trivial correlation coefficients. Moreover, these SCEE are E-stable. Numerical simulations suggest that E-stability also governs real-time learning.

Conversely, when beliefs depend on a constant then numerical evidence suggests that the only E-stable SCEE are those in which the correlation coefficients are zero. Our results are based on sufficient but not necessary conditions. It was argued that the assumption that F is increasing and symmetric prohibits non-trivial SCEE (when beliefs depend on a constant) from being E-stable. These results suggest (though they do not imply) that non-trivial SCEE occur in models with meaningful non-linearity. If the dynamics are well-represented by a linearization then the only SCEE are trivial ones. Furthermore, these results suggest that existence of non-trivial SCEE in which constant terms are present may depend on the non-monotonicity of the underlying law of motion as first presented by (Grandmont 1985) and emphasized in (Hommes, Sorger, and Wagener 2002). Future research should investigate these issues fully.

In future research we intend to examine existence and stability of SCEE in economic applications. In particular, we are interested in an extension to multivariate models such as the real business cycle model and the New-Keynesian monetary model. Both are models which are usually analyzed under linear approximations despite well-known interesting global phenomenon.

Appendix

This Appendix contains the proofs of most of the results of the paper.

Proof of Lemma 2 Let $s > s'$ and $I(z) = (-\infty, z]$. For real number w and set $A \subset \mathbb{R}$, define the notation

$$A - w = \{a - w | a \in A\}.$$

Note that $F' > 0$ implies $F(b^2 s) > F(b^2 s')$. Thus

$$\begin{aligned} P_b(s, I(z)) &= prob\{x_{t+1} \in I(z) | x_t = s\} \\ &= prob\{\eta_{t+1} \in I(z) - F(b^2 s)\} \\ &\leq prob\{\eta_{t+1} \in I(z) - F(b^2 s')\} \\ &= P_b(s', I(z)), \end{aligned}$$

where the inequality follows from the fact that $I(z) - F(b^2 s) \subset I(z) - F(b^2 s')$. ■

Proof of Lemma 3 Let $d = 0$ and $x_0 = a$. Let $x_b^* \geq 0$ be the largest point such that $F(b^2 x_b^*) = x_b^*$. Since x_b^* is a stable fixed point of the non-stochastic system $x_t = F(b^2 x_{t-1})$ there exists N such that if $\eta_t \leq 0$ for N consecutive times then $x_N \leq x_b^*$. If $x_b^* = 0$ we are done. If not let

$$h = \sup_{0 \leq x \leq x^*} F(x) - x,$$

and recall by assumption that $\overline{\eta} > h$. Let $h' = -1/2(h + \overline{\eta})$. Then $prob\{\eta_t < h'\} = \delta > 0$. Now notice that if $\eta_t < h'$ then

$$\begin{aligned} x_t - x_{t-1} &= F(b^2 x_{t-1}) - x_{t-1} + \eta_t \\ &\leq F(x_{t-1}) - x_{t-1} + \eta_t \\ &\leq h + \eta_t < h + h' = 1/2(h - \overline{\eta}) < 0. \end{aligned}$$

Thus if $x_N \leq x_b^*$ then there exists M so that if $\eta_t < h'$ M consecutive times then $x_{N+M} \leq 0$. By symmetry an analogous proof holds for $x_0 = -a$. ■

Proof of Lemma 5 It suffices to show the corresponding distribution functions converge pointwise; that is,

$$prob\{s' \in I(z) | s' = F(b_n^2 s_n) + \eta_t\} \to prob\{s' \in I(z) | s' = F(b^2 s) + \eta_t\}.$$

But this follows immediately from the fact that $F(b_n^2 s_n) \to F(b^2 s)$. Indeed

$$\begin{aligned} prob\{s' \in I(z) | s' = F(b_n^2 s_n) + \eta_t\} &= prob\{\eta_t \in I(z) - F(b_n^2 s_n)\} \\ &= \int_{-\infty}^{z - F(b_n^2 s_n)} f_\eta(x) dx, \end{aligned}$$

and this integral is continuous in its limits of integration. ■

Proof of Lemma 8 To prove statement one, it suffices to show that $E(x_t x_{t-1}) \geq 0$. But

$$\begin{aligned} E(x_t x_{t-1}) &= E(x_{t-1}F(b^2 x_{t-1})) \\ &= \int_{-\infty}^{\infty} xF(b^2 x) d\lambda_b(x) \geq 0 \end{aligned}$$

where the last inequality follows from the fact that the integrand is always positive. Statement 2 follows from the fact that $T(b)$ is a correlation coefficient. ■

Proof of Proposition 9 Let $T(b) = b$. We claim that this b together with the process generated by (4) is an SCEE. By definition $b = corr(x_t, x_{t-1})$, and so it remains to show that $E(x_t) = 0$. Recall that regardless of the initial distribution λ_0, convergence to λ_b obtains. Thus it suffices to show that if $E_{\lambda_0}(x) = 0$ then $E_{T^*\lambda_0}(x) = 0$. But

$$E_{T^*\lambda_0}(x) = E_{\lambda_0}(F(b^2 x)) = 0,$$

where the last equality follows from the fact that F is symmetric about zero. ■

Proof of Proposition 10 It suffices to show that if $b > 0$ then $T(b) < b$. Let $m = F'(0)$ and notice that $|mb^2 x| > |F(b^2 x)|$. Let λ be the unconditional distribution of x_t. Then

$$\begin{aligned} T(b) &= E(x_t^2)^{-1} E(x_t x_{t-1}) \\ &= E(x_t^2)^{-1} \int x_{t-1} F(b^2 x_{t-1}) d\lambda(x_{t-1}) \\ &\leq E(x_t^2)^{-1} \int mb^2 x_{t-1}^2 d\lambda(x_{t-1}) = mb^2 < b. \end{aligned}$$

■

Proof of Proposition 2.3 It suffices to prove there is a $b > 0$ so that $T(b) \geq b$. Pick $\hat{x}$ as in the premise of the proposition and let $m = F(\hat{x})/\hat{x}$ and $b = 1/m$. We claim this b works. First notice that $\hat{x} \leq x^*$: this follows from the fact that $x > x^* \Rightarrow F(x)/x < 1 \Rightarrow F(x)^2/x < F(x) < x < a$. This shows that $b \leq 1$ as well as that $F'(0) > 1$ thus distinguishing the premise of this proposition from that of the previous. Next notice that $b^2 a = \hat{x}^2 a / F(\hat{x})^2 < \hat{x}$. Thus $x \in [-a, a]$ implies $|b^2 x| < \hat{x}$. We claim this shows $|mb^2 x| < |F(b^2 x)|$ for $x \neq 0$. First consider $0 < x < \hat{x}$. Then the line $y = mx$ lies under the graph of $F(x)$; indeed, F is concave down for positive x and the line mx joins the origin and the point $(\hat{x}, F(\hat{x}))$. A symmetric argument holds for $-\hat{x} < x < 0$. An argument analogous to the one given to prove Proposition 10 shows that $T(b) \geq mb^2 = b$. ■

Proof of Lemma 15 Recall F is symmetric about α and $G(x) = F(x + \alpha)$ and, hence, is symmetric about the origin. If $G'(0) > 1$ let $x^* > 0$ be such that $G(x^*) = x^*$. Finally, let

$$h = sup_{0 \leq x \leq x^*} G(x) - x.$$

Recall that, by assumption, $h < \overline{\eta}$. Let $h' = -1/2(h + \overline{\eta})$ and let $prob\{\eta_t < h'\} = \delta > 0$. Now fix c and b thus defining a PLM and let $H(x) = F(c(1 + b) + b^2 x)$ so that the dynamic system is now defined by $x_t = H(x_{t-1}) + \eta_t$. We want to show the process x_t satisfies the mixing axiom. There are two cases.

Case 1: There exists exactly one point $\hat{x}$ so that $H(\hat{x}) = \hat{x}$. Then $\hat{x}$ is a globally stable fixed point of the non-stochastic system. Letting d be any point near $\hat{x}$ (for example $d = \hat{x} + \overline{\eta}/2$) suffices. Notice this argument holds is $\hat{x}$ is positive or negative.

Case 2: There exist points $\hat{x}_1 < \hat{x}_2 < \hat{x}_3$ so that $H(\hat{x}_i) = \hat{x}_i$. There are two symmetric subcases; we begin with $\hat{x}_2 > 0$. Pick $d = \hat{x}_1 + \overline{\eta}/2$. Both $\hat{x}_1$ and $\hat{x}_3$ are stable fixed points of the non-stochastic system, so reaching them from the nearest boundaries ($-a$ for $\hat{x}_1$ and a for $\hat{x}_3$) in finite time with positive probability is guaranteed. Also, our choice of d implies that if $\hat{x}_1$ is neared then d will be crossed with positive probability. Furthermore, if $x_t \le \hat{x}_2$ then the stability of $\hat{x}_1$ implies d will be crossed. We conclude that the only difficulty is guaranteeing x_t will move from $\hat{x}_3$ to $\hat{x}_2$ with positive probability. To see this, let $\hat{x}_2 \le x_{t-1} \le \hat{x}_3$. Notice that because $\hat{x}_2 > 0$, the graph of H is the graph of G scaled down by the b^2 term and shifted right. Indeed, letting $G^*(x) = G(b^2 x)$, we get that $H(x) = G^*(x + (c(1+b) - \alpha)/b^2)$. Thus, if $x \in (\hat{x}_2, \hat{x}_3)$ then $H(x) \le G(x)$. Then $\eta_t \le h'$ implies

$$\begin{aligned} x_t - x_{t-1} &= H(x_{t-1}) - x_{t-1} + \eta_t \\ &\le G(x_{t-1}) - x_{t-1} + \eta_t \\ &\le h + \eta_t < h + h' = 1/2(h - \overline{\eta}) < 0. \end{aligned}$$

Since with positive probability (δ) the movement left is bounded below we are done. The case $\hat{x}_2 \le 0$ is done analogously. ■

Proof of Lemma 3.1.2 That T_b is in $[-1,1]$ follows from the fact that it is a correlation coefficient; in fact, stationarity implies T_b is in $(-1,1)$. Now notice $|T_c| = |E(x_t)(1 - T_b)| \le 2|E(x_t)| \le 2a$. ■

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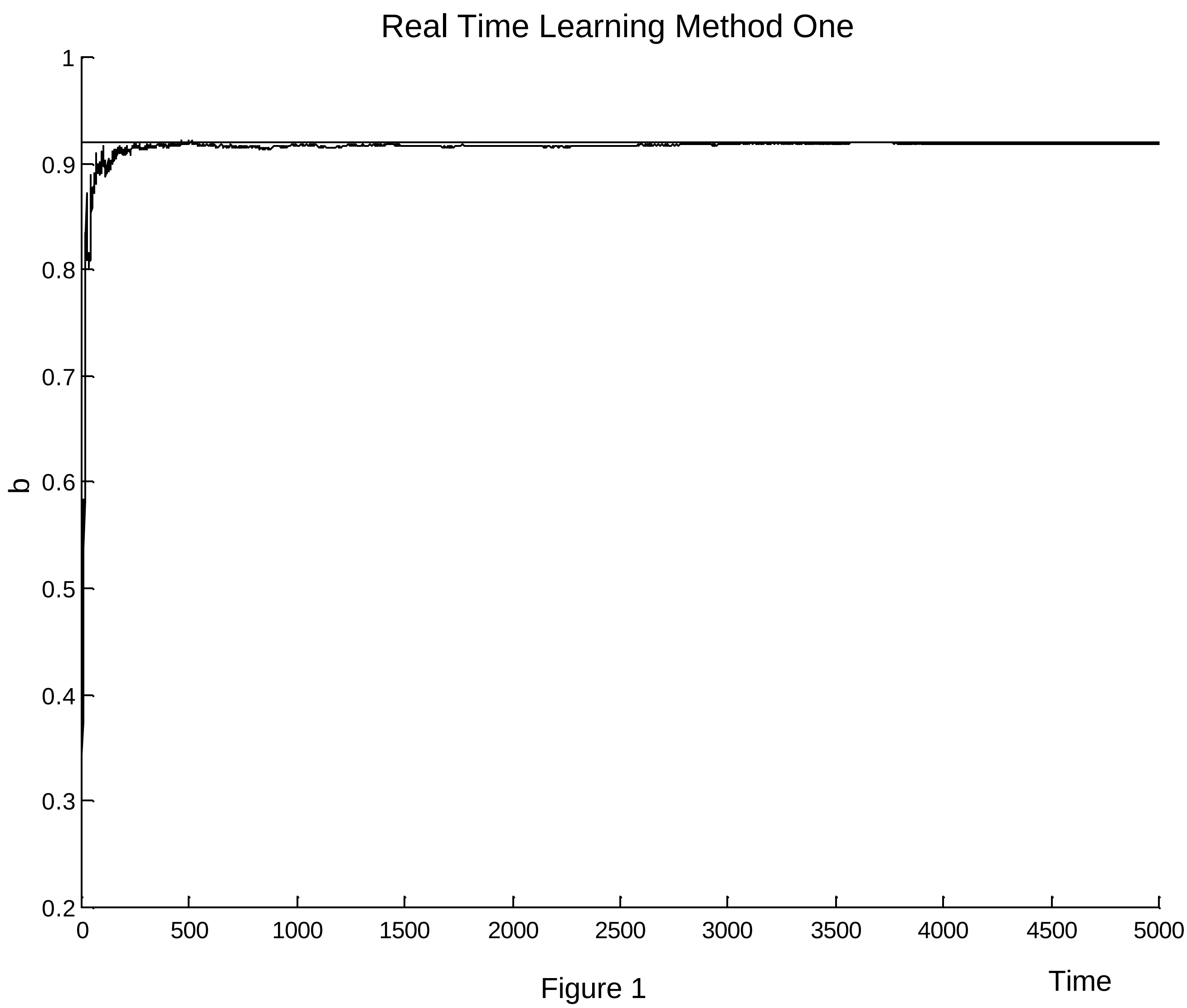
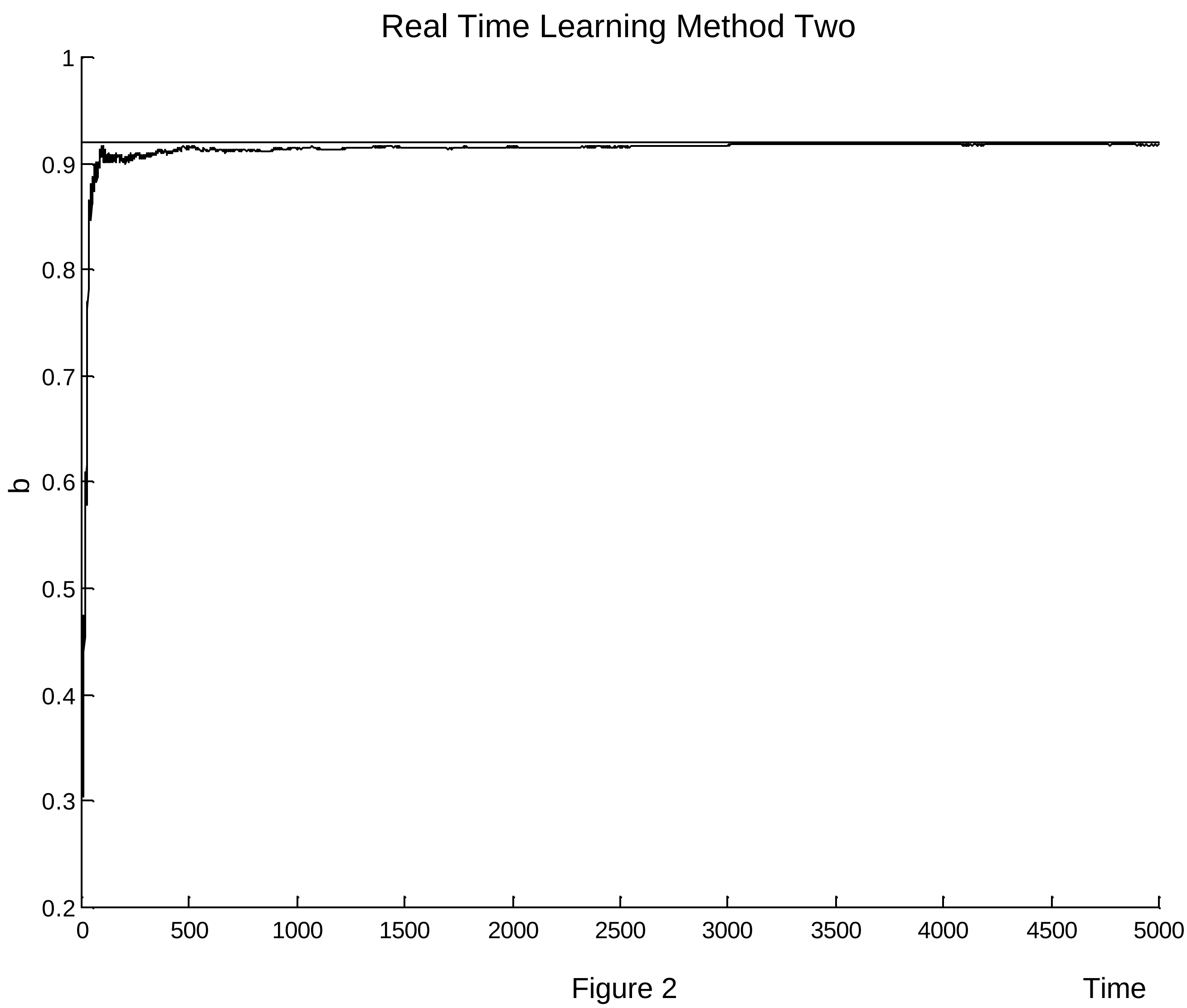

Real Time Learning Method One
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
b
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
Time

Figure 1



Real Time Learning Method Two
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
b
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
Time

Figure 2

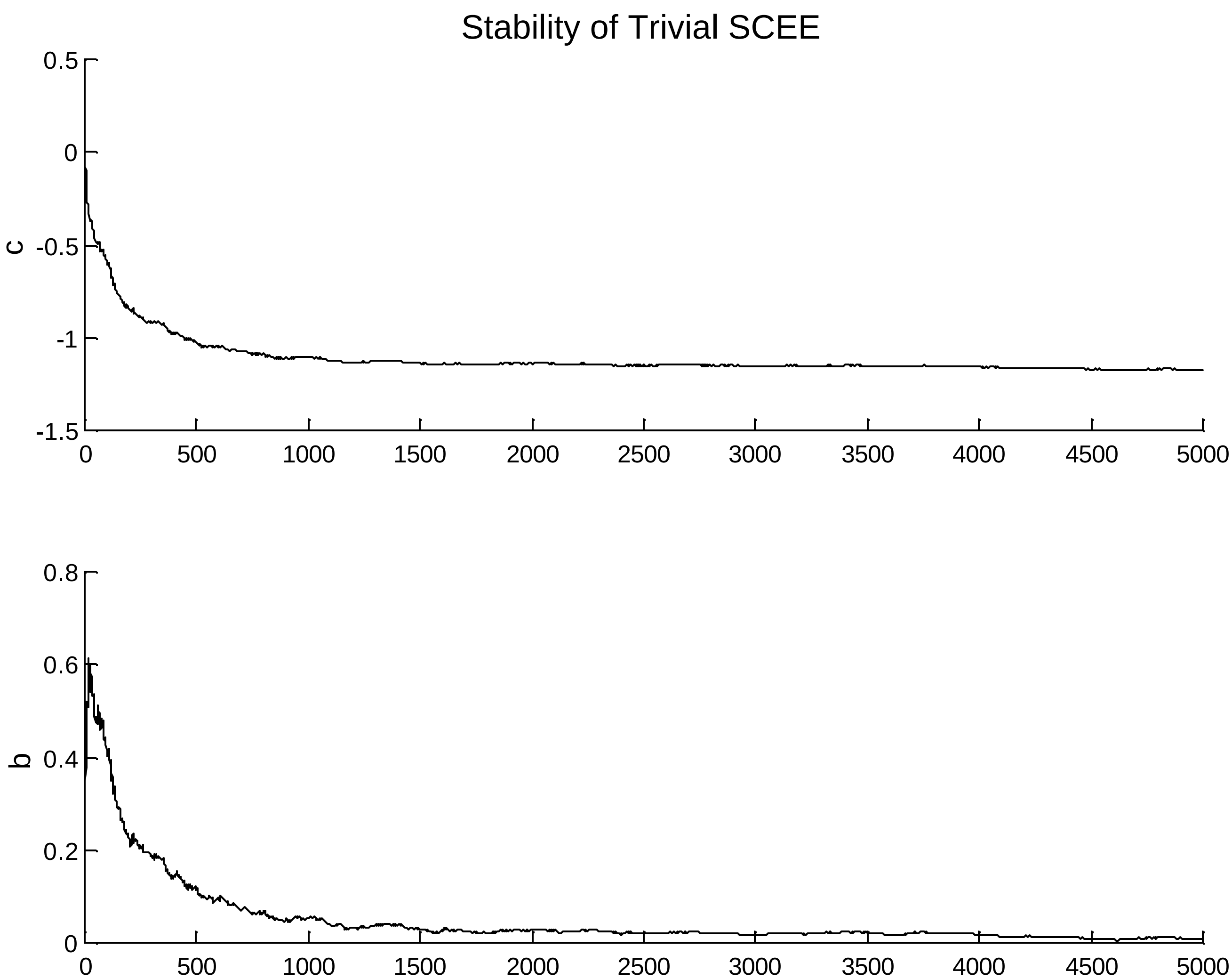

Stability of Trivial SCEE
c
0.5
0
-0.5
-1
-1.5
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
b
0.8
0.6
0.4
0.2
0
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

Figure 3

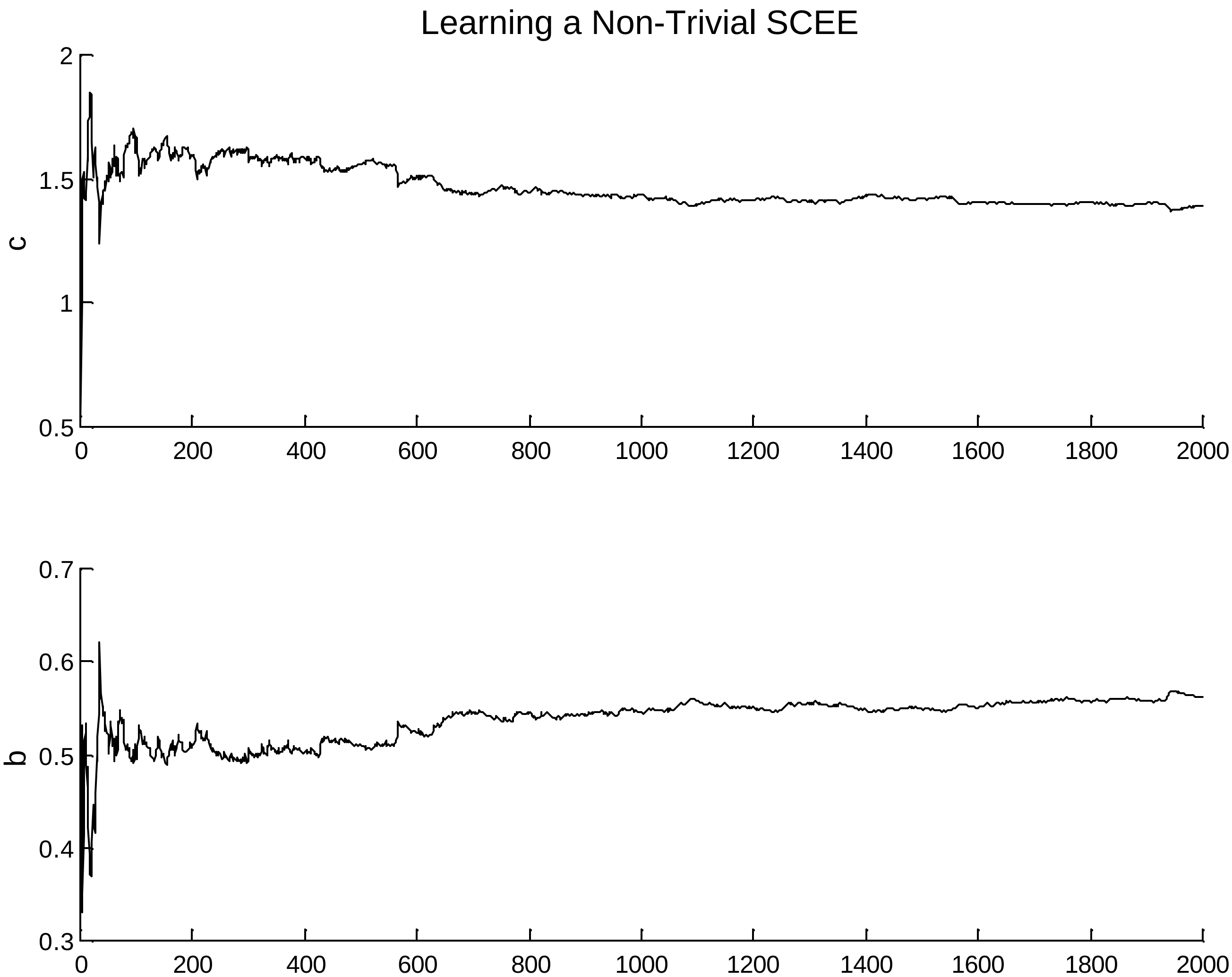

Learning a Non-Trivial SCEE
c
2
1.5
1
0.5
0
200
400
600
800
1000
1200
1400
1600
1800
2000
b
0.7
0.6
0.5
0.4
0.3
0
200
400
600
800
1000
1200
1400
1600
1800
2000

Figure 4

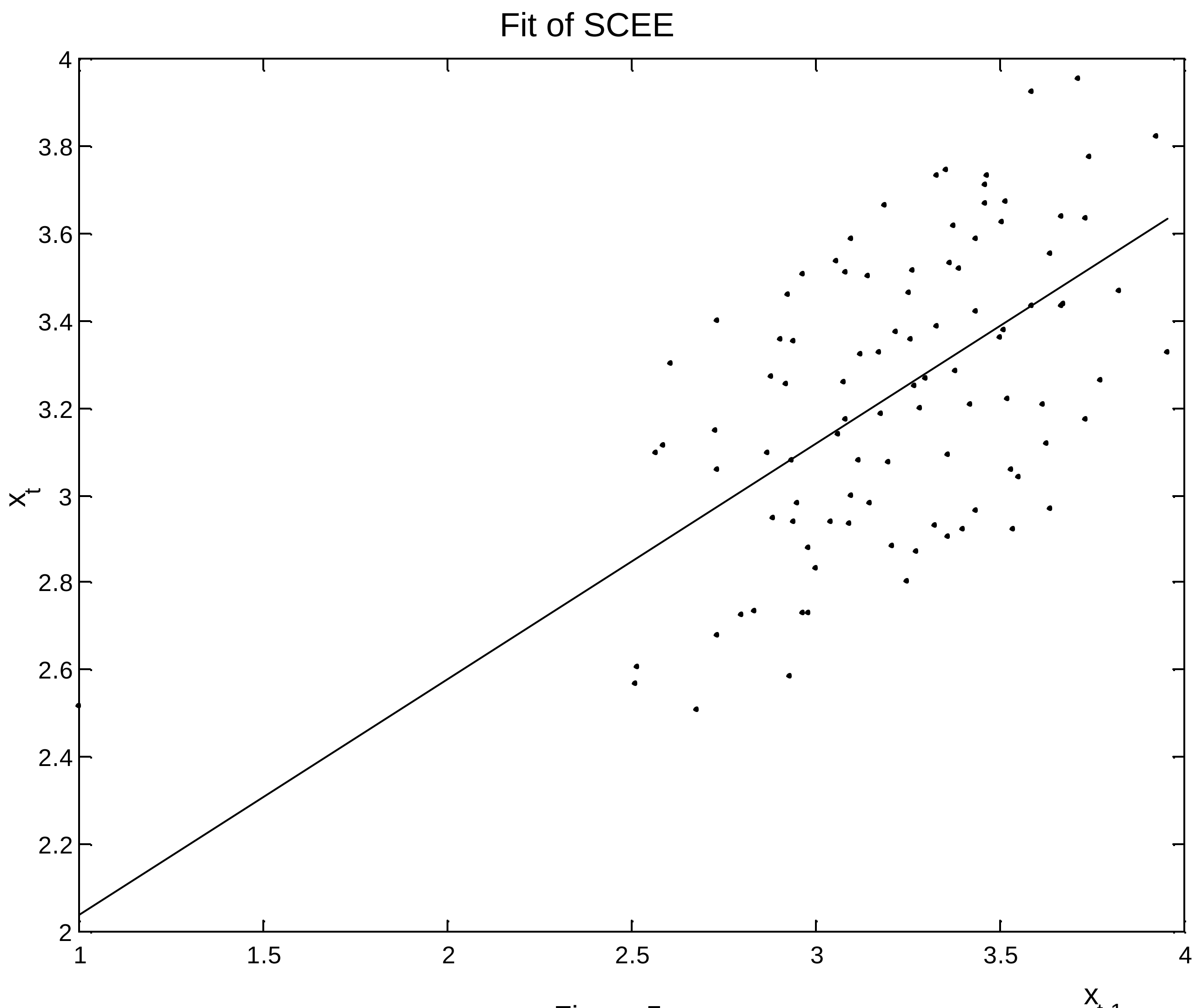


Figure 5