# Contraction Mappings

Mth 507 – Summer 1999 – Bent Petersen

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## 1 Introduction

I will assume some familiarity with topology and metric spaces. Thus the development will not be linear, that is, some concepts are mentioned before they are defined.

In the late 1800's set theory was developed largely by Cantor. During the same period Riemann, Ascoli, Arzela, Hadamard and others inspired by the needs of the calculus of variations introduced the concept of function spaces – sets in which the points are functions. Fréchet in his 1906 thesis used abstract set theory and introduced an axiomatic approach to function spaces. He considered abstract sets provided with an abstract limit process. He also introduced abstract metric spaces and raised questions of metrizability for topological spaces. He introduced and studied notions related to compactness, separability and completeness and discussed the relation between compactness and total boundedness. In his thesis he also applied his ideas to examples of function spaces; continuous functions on an interval, analytic functions in a domain. His thesis demonstrated that useful and interesting results could be obtained in an abstract setting.

The theory of abstract metric spaces was largely created by Fréchet and Hausdorff. Indeed the name *metric space* (metrischer Raum) seems to be due to Hausdorff. The familiar neighborhood formulation of topology is due largely to Hilbert, F. Riesz, Fréchet and especially Hausdorff. Hausdorff's 1914 book became a widely used standard text.

## 2 Metric Spaces

Let $X$ be a set. A *semimetric* on $X$ is a function

$$d\colon X \times X \to \mathbb{R}$$

such that

① $d(x,y) \geq 0, \;\; d(x,x) = 0$

② $d(x, y) = d(y, x)$

③ $d(x, y) \le d(x, z) + d(z, y)$

for each $x,\, y,\, z \in X$. If in addition we require $d(x, y) = 0$ implies $x = y$ then $d$ is called a *metric*. A *semimetric space* (respectively, a *metric space* is a set $X$ together with a semimetric (respectively, a metric) $d$ on $X$. Most of the results obtained for metric spaces apply equally well to semimetric spaces, though some care is needed, as, for example, a compact set in a semimetric space need not be closed, limits of convergent sequences need not be unique, and so forth.

The most familiar example of a metric space is the set of real numbers $\mathbb{R}$ with the usual metric

$$d(x, y) = |x - y|, \quad x,\, y \in \mathbb{R}.$$

Another useful metric on $\mathbb{R}$ is given by

$$d(x, y) = |\arctan(x) - \arctan(y)|.$$

Given $x \in X$ and $\epsilon > 0$ we define the *open $\epsilon$–ball at $x$* to be

$$W(x, \epsilon) = \{\, y \in X \;\mid\; d(x, y) < \epsilon \,\}\,.$$

A subset $U$ of $X$ is said to *open* if for each $x \in U$ there is $\epsilon > 0$ such that $W(x, \epsilon) \subseteq U$. Note that the empty set $\varnothing$ and the whole space $X$ are open. A subset $V$ of $X$ is said to be a *neighborhood* of $x$ if $W(x, \epsilon) \subseteq V$ for some $\epsilon > 0$. A subset $A$ of $X$ is *closed* if its complement is open. An example of a closed set is the *closed $\epsilon$–ball at $x$*

$$B(x, \epsilon) = \{\, y \in X \;\mid\; d(x, y) \le \epsilon \,\}\,.$$

**Exercise 1.** Show that $B(x, \epsilon)$ is closed. Give an example where the closure of $W(x, \epsilon)$ is a proper subset of $B(x, \epsilon)$. Show that $W(x, \epsilon)$ is open.

Two important notions introduced early in the study of metric spaces are convergence of sequences and continuity of functions. Initially these concepts are given a metric formulation, but they are usually quickly rephrased in terms of neighborhoods and open sets.

Let $(x_n)_{n\ge1}$ be a sequence in the metric space $X$. We say that $(x_n)_{n\ge1}$ *converges* to $x \in X$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x$$

if for each $\epsilon > 0$ there is $N$ such that

$$d(x, x_n) < \epsilon \text{ if } n \ge N.$$

We can formulate the notion of convergence entirely in terms of neighborhoods with no direct reference to the metric as follows: $x_n \to x$ if and only if for each neighborhood $V$ of $x$ there exists $N$ such that $x_n \in V$ for each $n \ge N$.

Let $X$ and $Y$ be metric spaces and let $f : X \to Y$. Let $a \in X$. We say that $f$ is *continuous at* $a$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $x \in X,\;\; d(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$. We say $f$ is *continuous on* $X$ or simply *continuous* if $f$ is continuous at each point of $X$.

**Exercise 2.** The notion of continuity at a point may be formulated in terms of neighborhoods: $f$ is continuous at $a$ if and only if for each neighborhood $V$ of $f(a)$ there is a neighborhood $W$ of $a$ such that $f(W) \subseteq V$.

**Exercise 3.** The notion of continuity may be formulated in terms of open sets: $f$ is continuous if and only if for each open set $V \subseteq Y$ the set $f^{-1}(V)$ is an open subset of $X$.

Let $(x_n)_{n\geq 1}$ be a sequence in the metric space $X$ (with metric $d$). We call $(x_n)_{n\geq 1}$ a *Cauchy sequence* if for each $\epsilon > 0$ there exists $N$ such that $d(x_m, x_n) < \epsilon$ if $m, n \geq N$. If each Cauchy sequence converges then we say that $X$ is *complete*.

Completeness is a metric property – it is not determined by the underlying topology. The concept can be extended to the class of *uniform spaces* but we also have to deal with a more general concept of convergence then – convergence of nets or filters.

**Exercise 4.** The set of real numbers $\mathbb{R}$ is complete with respect to the usual metric

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R},$$

but not with respect to the metric

$$d(x, y) = |\arctan(x) - \arctan(y)|.$$

Both metrics induce the same topology, in the sense that they have the same open sets.

**Exercise 5.** Let $(x_n)_{n\geq 1}$ be a convergent sequence on the metric space $X$. Then $(x_n)_{n\geq 1}$ is a Cauchy sequence.

**Exercise 6.** In a metric space if a subsequence of a Cauchy sequence converges then the original sequence converges. It follows that a compact metric space is complete.

**Exercise 7.** Let $X$ be a complete metric space and let $Y \subseteq X$. If $Y$ is closed then $Y$ is complete. Conversely if $Y$ is complete then $Y$ is closed.

# 3 Contraction Mapping Principle

Let $X$ be a metric space. A mapping $T : X \to X$ is called *contraction map* if there exists a constant $c$ with $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c\, d(x, y), \quad x, y \in X.$$

The constant $c$ is called the *contractivity coefficient*.

**Theorem 1 (Contraction Mapping Principle).** *Let $X$ be a complete metric space and let $T : X \to X$ be a contraction mapping with contractivity coefficient $c$. Let $x_0 \in X$ and inductively define*

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

*The $T$ has a unique fixed point $a$, the sequence $x_n$ converges to $a$ and*

$$d(a, x_n) \leq c^n\, d(a, x_0).$$

*Proof.* If $a$ and $b$ are fixed points then $d(a,b) = d(T(a),T(b)) \leq c\,d(a,b)$ which implies $d(a,b) = 0$ since $0 \leq c < 1$. If $a$ is a fixed point of $T$, that is, $T(a) = a$ then $d(a,x_n) = d(T(a),T(x_{n-1}) \leq c\,d(a,x_{n-1})$. By induction we obtain $d(a,x_n) \leq c^n\,d(a,x_0)$. If $n \geq 1$ then $d(x_n,x_{n+1}) = d(T(x_{n-1},T(x_n)) \leq c\,d(x_{n-1},x_n)$ and so by induction $d(x_n,x_{n+1}) \leq c^n\,d(x_0,x_1)$. It follows if $0 \leq n < m$ then

$$d(x_n,x_m) \leq \left(c^n + \cdots + c^{m-1}\right)\,d(x_0,x_1) \leq \frac{c^n}{1-c}\,d(x_0,x_1)$$

and therefore $(x_n)$ is a Cauchy sequence. Since $X$ is complete this sequence converges to a point $a$. Now $T(a) = a$ by continuity of $T$. □

Note taking the limit as $m \to \infty$ in the inequality in the proof of the theorem we obtain

$$d(a,x_n) \leq \frac{c^n}{1-c}\,d(x_0,x_1)$$

which is sometimes more convenient than the inequality stated in the theorem.

The contraction mapping principle is useful, but the hypotheses are very strong and may be difficult to satisfy. If we examine the proof of the contraction mapping principle we are led to a simple result which may be more useful.

**Theorem 2.** *Let $X$ be a complete metric space and let $T : X \to X$ be a continuous map. Let $x_0 \in X$ and inductively define*

$$x_{n+1} = T(x_n).$$

*If*

$$\sum_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty$$

*then the sequence $(x_n)_{n\geq 0}$ converges to a fixed point $a$ of $T$. Moreover*

$$d(a,x_n) \leq \sum_{k=n}^{\infty} d(x_k,x_{k+1}).$$

*Proof.* By the triangle inequality if $0 \leq n < m$ then

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}).$$

Thus $(x_n)_{n\geq 0}$ is a Cauchy sequence. If the limit is $a$ then by continuity of $T$ we have $a = T(a)$. Taking the limit as $m \to \infty$ in the inequality above we obtain the required estimate. □

## 4 Example: Ordinary Differential Equations

In this section we show how Contraction Mapping Principle implies a version of the fundamental existence and uniqueness theorem for Cauchy initial value problems for ordinary differential equations.

Let $\Omega$ be an open subset of the plane and let $f$ be a continuous function in $\Omega$. Let $(a, c) \in \Omega$ and consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \;\; y(a) = c.$$

By integrating we see this initial value problem is equivalent to the integral equation

$$y(x) = c + \int_a^x f(t, y(t))\, dt,$$

that is, we are looking for a fixed point for the map defined by the integral. Let $p > 0$ and $q > 0$ be chosen so that the rectangle $D$ defined by

$$0 \leq x - a \leq p, \quad |\, y - c \,| \leq q$$

is contained in $\Omega$. Let $M$ be an upper bound for $|\, f \,|$ on the rectangle $D$. Choose $h$ with

$$0 < h \leq \min(p, q/M).$$

Let $b = a + h$ and let

$$X_h = \{\, u \in \mathrm{C}([a, b]) \;\;|\;\; u(a) = c, \;\; |\, u(x) - c \,| \leq q \text{ if } a \leq x \leq b \,\}\,.$$

If $u \in X_h$ define $Tu$ by

$$Tu(x) = c + \int_a^x f(t, u(t))\, dt.$$

Clearly

$$|\, Tu(x) - c \,| \leq M(x - a) \leq Mh \leq q$$

and so $T : X_h \to X_h$. The space $\mathrm{C}([a, b])$ is a complete metric space when provided with the metric $d(u, v) = \sup |\, u(t) - v(t) \,|\,,\; a \leq t \leq b$ and $X_h$ is a closed subset. Thus $X_h$ is a complete metric space. Now suppose that $f$ is Lipschitz continuous in the second variable in the rectangle $D$, say

$$|\, f(x, y) - f(x, z) \,| \leq L \;|\, y - z \,|, \qquad \text{for } (x, y), (x, z) \in D.$$

Let $0 < \delta < 1$. If $u, v \in X_h$ then

$$|\, Tu(x) - Tv(x) \,| \leq \int_a^x |\, f(t, u(t)) - f(t, v(t)) \,|\; dt$$

$$\leq \int_a^x L \;|\, u(t) - v(t) \,|\; dt \leq Lh\, d(u, v).$$

Thus $T$ is continuous. Moreover if $Lh \leq \delta$ then this inequality implies that $T$ is a contraction mapping and so has a unique fixed point in $X_h$. A simple estimate shows that any solution of the initial value problem must be in $X_h$. Thus we have the existence of a unique solution to the initial value problem on the interval $[a, b]$ where $b = a + h$ and

$$h = \min(p, q/M, \delta/L).$$

The iterates $u_{n+1} = T(u_n)$ converge rapidly to the fixed point. This method of approximating the solution was published by Liouville as early as 1838, though the general method is usually credited to Picard, 1890. The successive approximants are called *Picard iterates*.

We can do better. Let $y_0(x) = c$ and inductively define $y_{n+1} = Ty_n$. Then

$$|\, y_1(x) - y_0(x)\,| \leq \int_a^x |\, f(t, y_0(t))\,|\; dt \leq M\,(x-a)$$

and

$$\begin{aligned} |\, y_2(x) - y_1(x)\,| = |\, Ty_1(x) - Ty_0(x)\,| &\leq \int_a^x L\;|\, y_1(t) - y_0(t)\,|\; dt \\ &\leq \int_a^x LM\,(t-a)\,dt = \frac{LM(x-a)^2}{2}. \end{aligned}$$

By induction we see

$$|\, y_n(x) - y_{n-1}(x)\,| \leq \frac{L^{n-1}M(x-a)^n}{n\,!}$$

and so

$$d(y_n, y_{n-1}) \leq \frac{M}{L}\,\frac{(Lh)^n}{n\,!}.$$

It follows that the sequence $(y_n)_{n\geq 0}$ converges to a fixed point of $T$. Thus we have existence of a solution to the initial value problem on the interval $[a, b]$ where $b = a + h$ and

$$h = \min(p, q/M).$$

This little argument shows that there is some value to extending the contraction mapping principle as we did above.

In the present case uniqueness requires an additional argument, which may be found in standard texts on ordinary differential equations; for example, Nemytskii and Stepanov, 1960, or Coddington and Levinson 1955. Finally note that the argument is essentially unchanged if $y$ and $f$ are vector valued. Since any system of differential equations can be replaced by a system of first order equations, we obtain existence and uniqueness under mild hypotheses for the Cauchy initial value problem for any system of differential equations (in normal form).

## 5 Example: Iterated Function Systems

In this section we describe an application of the Contraction Mapping Principle to the study of iterated function systems and their corresponding fractals.

Let $X$ be a metric space. If $B$ is a nonempty subset of $X$ we define

$$d(x, B) = \inf_{y \in B} d(x, y).$$

Let $x, y \in X$. If $\epsilon > 0$ choose $z \in B$ such that $d(y, z) < d(y, B) + \epsilon$. Then $d(x, B) \leq d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + d(y, B) + \epsilon$. Thus we conclude

$$|d(x, B) - d(y, B)| \leq d(x, y), \quad x, y \in X.$$

It follows that $x \longrightarrow d(x, B)$ is a continuous nonnegative function vanishing precisely on $\overline{B}$.

Now if $A$ and $B$ are nonempty subsets of $X$ we define

$$d(A, B) = \sup_{x \in A} d(x, B).$$

Obviously $0 \leq d(A, B) \leq \infty$. Let $A, B, C$ be nonempty subsets of $X$. Let $x \in A$ and let $\epsilon > 0$. Choose $z \in C$ such that $d(x, z) < d(x, C) + \epsilon$. Now $d(x, B) \leq d(x, z) + d(z, B) \leq d(x, C) + d(z, B) + \epsilon$. SInce $z \in C$ we have $d(z, B) \leq d(C, B)$ and therefore $d(x, B) \leq d(x, C) + d(C, B)$. Taking the supremum over $x \in A$ we obtain

$$d(A, B) \leq d(A, C) + d(C, B).$$

We have to be careful with the order here:

**Exercise 8.** Find an example with $d(A, B) \neq d(B, A)$.

Let $\mathfrak{K}(X)$ be the set of nonempty compact subsets of $X$. If $A, B \in \mathfrak{K}(X)$ we define the *Hausdorff metric* $d_H$ by

$$d_H(A, B) = \max \left\{ d(A, B),\ d(B, A) \right\}.$$

If $A$ is a subset of $X$ we define the *closed $\epsilon$–neighborhood* $A_\epsilon$ of $A$ by

$$A_\epsilon = \{\, y \in X \ \mid \ d(x, y) \leq \epsilon \text{ for some } x \in A \,\}.$$

**Exercise 9.** If $A, B \in \mathfrak{K}(X)$ and $\epsilon > 0$ then $d(A, B) \leq \epsilon$ if and only if $A \subseteq B_\epsilon$.

**Exercise 10.** $d_H$ is a metric on $\mathfrak{K}(X)$.

**Theorem 3 (Completeness of $\mathfrak{K}(X)$).** *If $X$ is a complete metric space then $\mathfrak{K}(X)$ is complete in the Hausdorff metric. If $(A_n)_{n \geq 1}$ is a Cauchy sequence in $\mathfrak{K}(X)$ and $A = \lim_{n \to \infty} A_n$ then*

$$A = \left\{ x \in X \ \middle| \ \text{there exists } x_n \in A_n \text{ so that } \lim_{n \to \infty} x_n = x \right\}.$$

I'll leave the proof as an exercise.

Now let $T_j$ be contraction mappings of $X$ with contractivity coefficients $c_j$, $j = 1, \cdots n$. Define $T : \mathfrak{K}(X) \to \mathfrak{K}(X)$ by

$$T(A) = T_1(A) \bigcup \cdots \bigcup T_n(A), \quad A \in \mathfrak{K}(X).$$

One can show that $T$ is a contraction mapping with contractivity coefficient $c = \max_j c_j$. Thus $T$ has a unique fixed point $A$ in $\mathfrak{K}(X)$ and we can approximate it by computing iterates of $T$ applied to an arbitrary element of $\mathfrak{K}(X)$. The system $(T_j)_{1 \leq j \leq n}$ is called an *iterated function system*, abbreviated as IFS. The fixed point $A$ is called the *fractal* determined by the IFS. Note we can approximate the fixed point $A$ quite rapidly by computing the iterates $T^n(F)$ where $F$ is some initial arbitrary finite set.

**Exercise 11.** Let $X = [0, 1]$ and consider the IFS $\{T_1, T_2\}$ defined by $T_1(t) = t/3$ and $T_2(t) = 2/3 + t/3$. Show by a direct calculation that the Cantor set is the fixed point of this IFS.

We may view a compact in the plane as a "picture." If it happens to be the fixed point of an IFS then the handful of real parameters which describe the IFS may be viewed as a very efficient encoding of the picture. Now the inverse IFS problem – given the fixed point find a corresponding IFS – is clearly of some interest.



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