

Prime Number Theorem

Mth 619

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1 Gauss

Let $\pi(x)$ be the number of primes $p \leq x$. In 1792 or 1793 Gauss discovered *empirically*

$$\pi(x) \approx \int \frac{dx}{\log x}.$$

Here \approx means approximate equality. Gauss described his work in a letter to Enke in 1849, (Gauss [7], volume 2, page 444).

Note Gauss does not explicitly indicate the limits of integration and some authors assume he meant the lower limit to be 2. In his letter Gauss includes an excerpt of his prime number counts (up to $x = 3,000,000$) and the corresponding estimates. From the estimates Gauss gives it is clear that Gauss actually had in mind the logarithmic integral $\text{Li}(x)$ given by the principal value integral

$$\text{Li}(x) = (\text{Pv}) \int_0^x \frac{dt}{\log t}.$$

(See below for some discussion of $\text{Li}(x)$).

There are some small errors in Gauss's table. Here is a corrected version (partly checked by Maple 6).

x	$\pi(x)$	$\text{Li}(x)$	Error
500,000	41,538	41,606.29	-68.29
1,000,000	78,498	78,627.55	-129.55
1,500,000	114,155	114,263.09	-108.09
2,000,000	148,933	149,054.83	-121.83
2,500,000	183,072	183,244.99	-172.99
3,000,000	216,816	216,970.56	-154.56

The errors are astonishingly small. Note it appears that $\text{Li}(x)$ overestimates $\pi(x)$, however, this has been shown not to be the case, for some very large (but unknown) x .

To get a better idea of how well $\text{Li}(x)$ approximates $\pi(x)$ let us look at a sample of relative errors:

x	$\pi(x)$	Rel. Error in $\text{Li}(x)$
10^2	25	-0.2050456633632
10^3	168	-0.0572003451795
10^4	1,229	-0.0139440324650
10^5	9,592	-0.0039417223781
10^6	78,498	-0.0016503498110
10^7	664,579	-0.0005107068514
10^8	5,761,455	-0.0001309348850
10^9	50,847,534	-0.0000334521041
10^{10}	455,052,511	-0.0000068202824
10^{11}	4,118,054,813	-0.0000028138580
10^{12}	37,607,912,018	-0.0000010174137
10^{13}	346,065,536,839	-0.0000003148856
10^{14}	3,204,941,750,802	-0.0000000982514
10^{15}	29,844,570,422,669	-0.0000000352700
10^{16}	279,238,341,033,925	-0.0000000115121
10^{17}	2,623,557,157,654,233	-0.0000000030327
10^{18}	24,739,954,287,740,860	-0.0000000008872

Again we see the errors are breath-takingly small.

The round-off error in computing the relative errors above can be quite substantial. Before using this table for anything you ought to check it. I used Maple 6 at very high precision to produce this table.

2 Legendre

Legendre in 1798 published the conjecture

$$\pi(x) \approx \frac{x}{A \log x + B}$$

and in 1808 he asserted

$$\pi(x) = \frac{x}{\log x - A(x)}, \quad A(x) \approx 1.08366$$

In his letter to Enke (1849) Gauss discusses Legendre's formula and computes the values of $A(x)$ corresponding to the prime counts in the table above.

x	$\pi(x)$	$A(x)$
500,000	41,538	1.085192
1,000,000	78,498	1.076333
1,500,000	114,155	1.080947
2,000,000	148,933	1.079801
2,500,000	183,072	1.075972
3,000,000	216,816	1.077506

Note this list differs slightly from the one given by Gauss because I have used the (hopefully) corrected prime counts.

Gauss now goes on to say that it appears the (average) value of $A(x)$ decreases. He does not say what he means by the average value, but whatever it is, if it decreases it certainly has a limit. He says that he dares not conjecture whether the limit is 1, or some number different from 1, or if there is a simple limit at all. One can not help but wonder though why he mentions the number 1 at all.

Nowadays we have larger prime counts available, so we have an opportunity to check Gauss's intuition concerning $A(x)$.

x	$\pi(x)$	$A(x)$
10^2	25	0.60517018
10^3	168	0.95537433
10^4	1,229	1.07364387
10^5	9,592	1.08757100
10^6	78,498	1.07633249
10^7	664,579	1.07097559
10^8	5,761,455	1.06395401
10^9	50,847,534	1.05662871
10^{10}	455,052,511	1.05036512
10^{11}	4,118,054,813	1.04512642
10^{12}	37,607,912,018	1.04087170
10^{13}	346,065,536,839	1.03734543
10^{14}	3,204,941,750,802	1.03437618
10^{15}	29,844,570,422,669	1.03184412
10^{16}	279,238,341,033,925	1.02966041
10^{17}	2,623,557,157,654,233	1.02775776
10^{18}	24,739,954,287,740,860	1.02608512

This table certainly suggests that $A(x)$ tends to 1, or close to it, perhaps in some average sense, but the graph of $A(x)$ is pretty chaotic looking so who knows.

We can however use the table to formulate a conjecture concerning the size (or accuracy) of the tables of primes available to Legendre ($A(x) = 1.08366\dots$)!

The large prime counts in the table above are taken from Reisel [13]. The smaller counts were computed using Maple 6, and revealed some unfortunate errors in Maple 6.

x	$\pi(x)$	Maple 6 $\pi(x)$	Note
10^6	78,498	78,498	
10^7	664,579	644,579	oops
10^8	5,761,455	5,761,455	
10^9	50,847,534	60,847,534	oops
10^{10}	455,052,511	455,052,511	

These errors are particularly troublesome since the numbers “look” right.

3 The Logarithmic Integral

The logarithmic integral $\text{Li}(x)$, $x > 1$, is defined as the Cauchy principal value of the divergent integral $\int_0^\infty dt/\log t$. Explicitly

$$\begin{aligned} \text{Li}(x) &= \lim_{\epsilon \rightarrow 0} \left\{ \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right\} \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \int_0^{1-\epsilon} \left(\frac{1}{\log t} - \frac{1}{t-1} \right) dt + \log \epsilon \right. \\ &\quad \left. + \int_{1+\epsilon}^x \left(\frac{1}{\log t} - \frac{1}{t-1} \right) dt + \log(x-1) - \log \epsilon \right\} \\ &= \log(x-1) + \int_0^x \left(\frac{1}{\log t} - \frac{1}{t-1} \right) dt. \end{aligned}$$

This last integrand is $\frac{1}{2} - \frac{1}{12}(t-1) + \mathcal{O}((t-1)^2)$ and so is integrable on $(0, x)$.

The last expression here shows $\text{Li}(x) \rightarrow -\infty$ as $x \rightarrow 1$ and shows the derivative is $\frac{1}{\log x}$ so $\text{Li}(x)$ is increasing on $(1, \infty)$. Since $\text{Li}(x)$ is positive for large x we see it has a unique root μ . One can show $\mu = 1.45136923488338105\dots$. We can avoid the singularity in the integrand by noting that for any $\nu > 1$ we have

$$\text{Li}(x) = \text{Li}(\nu) + \int_\nu^x \frac{dt}{\log t}$$

and therefore

$$\text{Li}(x) = \int_{\mu}^x \frac{dt}{\log t}.$$

Or, setting $\nu = 2$ we have the convenient

$$\text{Li}(x) = 1.045163780 \cdots + \int_2^x \frac{dt}{\log t}.$$

Proposition 3.1.

$$\text{Li}(x) = \sum_{k=1}^n (k-1)! \frac{x}{(\log x)^k} + \mathfrak{o}\left(\frac{x}{(\log x)^n}\right).$$

Proof. If we integrate by parts n times we have

$$\text{Li}(x) = \frac{x}{\log x} + \cdots + (n-1)! \frac{x}{(\log x)^n} + c_n + n! \int_2^x \frac{dt}{(\log t)^{n+1}}$$

where c_n is a constant. It now suffices to show

$$\lim_{x \rightarrow \infty} \frac{(\log x)^n}{x} \left(c_n + n! \int_2^x \frac{dt}{(\log t)^{n+1}} \right) = 0.$$

Dividing the interval of integration at $x^{1/2}$ we have

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{(\log x)^n}{x} \int_2^x \frac{dt}{(\log t)^{n+1}} \\ & \leq \frac{(\log x)^n}{x} \frac{x^{1/2} - 2}{(\log 2)^{n+1}} + \frac{(\log x)^n}{x} \frac{x - x^{1/2}}{(\log x)^{n+1}} 2^{n+1} \\ & \leq \left(\frac{\log x}{x^{1/2n}} \right)^n + \frac{2^{n+1}}{\log x} \end{aligned}$$

which converges to 0 as $x \rightarrow \infty$. □

The idea of dividing the interval of integration at $x^{1/2}$ is a standard trick. See Edwards [6], page 85.

The notation

$$f(x) \sim g(x), \quad x \rightarrow a$$

means

$$\lim_{x \rightarrow a} f(x)/g(x) = 1, \quad \text{that is} \quad f(x) = g(x) + \mathfrak{o}(g(x)), \quad x \rightarrow a.$$

We say $f(x)$ is asymptotically equal to $g(x)$ as $x \rightarrow a$. We frequently omit the expression “ $x \rightarrow a$ ”, especially if $a = \infty$ or if a may be inferred from the context.

From the proposition above with $n = 1$ we have

$$\text{Li}(x) = \frac{x}{\log x} + \mathfrak{o}\left(\frac{x}{\log x}\right)$$

or

$$\text{Li}(x) \sim \frac{x}{\log x}.$$

4 Prime Number Theorem

The prime number theorem is

$$\pi(x) \sim \text{Li}(x), \quad x \rightarrow \infty.$$

In view of the asymptotic properties of $\text{Li}(x)$ an equivalent statement is

$$\pi(x) \sim \frac{x}{\log x}.$$

The statement of the prime number theorem is often attributed to Gauss but it is not clear from his letter to Enke whether or not he had asymptotic equality in mind. He does express his belief that ultimately $\text{Li}(x)$ provides a better approximation than Legendre’s formula, but this statement emphasizes finite approximation rather than asymptotic equality.

Note if $A(x)$ is bounded (as it appears to be) then

$$\frac{x}{\log x - A(x)} \sim \frac{x}{\log x}$$

so from an asymptotic point of view there is no difference between the various formulae.

5 Čebyšev

In 1850 Čebyšev [3] proved a result far weaker than the prime number theorem — that for certain constants $0 < A_1 < 1 < A_2$

$$A_1 < \frac{\pi(x)}{x/\log x} < A_2.$$

Moreover he obtained estimates for the constants $0 < A_1 < 1 < A_2$. An elementary proof of Čebyšev's theorem is given in Andrews [1]. Čebyšev introduced the functions

$$\begin{aligned}\theta(x) &= \sum_{p \leq x, p \text{ prime}} \log p, && (\check{\text{C}}\text{ebyšev theta function}) \\ \psi(x) &= \sum_{p^n \leq x, p \text{ prime}} \log p, && (\check{\text{C}}\text{ebyšev psi function}).\end{aligned}$$

Note

$$\psi(x) = \sum_{n=1}^{\infty} \theta(x^{1/n})$$

where the sum is finite for each x since $\theta(x^{1/n}) = 0$ if $x < 2^n$. Čebyšev proved that the prime number theorem is equivalent to either of the relations

$$\theta(x) \sim x, \quad \psi(x) \sim x.$$

In addition Čebyšev showed that if $\lim_{x \rightarrow \infty} \frac{\theta(x)}{x}$ exists, then it must be 1, which then implies the prime number theorem. He was, however, unable to establish the existence of the limit.

6 Dirichlet

In 1837 Dirichlet published the proof of a famous conjecture of Legendre, namely if h and k are relatively prime positive integers then the sequence

$$h, h + k, h + 2k, \dots, h + nk, \dots$$

contains infinitely many primes. Such a sequence is called an arithmetical progression. A number of special cases were known but Dirichlet proved the general case. Perhaps what surprised many people is that Dirichlet used continuous methods, that is, real analysis.

7 Riemann

Like Gauss, Riemann in 1859 formulated his estimate of $\pi(x)$ in terms of the logarithmic integral $\text{Li}(x)$. Like Dirichlet, Riemann used continuous methods in his arguments, but in this case, functions of a *complex* variable. The remarkable

results obtained by Dirichlet and Riemann guaranteed that analytic methods would be important in number theory.

In his justly famous 1859 paper [14] Riemann related the relative error in the asymptotic approximation

$$\pi(x) \sim \text{Li}(x)$$

to the distribution of the *complex* zeros of the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

The Riemann zeta function was actually introduced by Euler as early as 1737. It was used of by Čebyšev (in the real domain) prior to Riemann's use of it. Euler also discovered the *functional equation*

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (1)$$

which he published in 1749. The functional equation was proved by Riemann in [14].

Riemann does not prove the prime number theorem in his 1859 paper. His object was to find an explicit analytic expression for $\pi(x)$, and he does so. He does comment that $\pi(x)$ is about $\text{Li}(x)$ and that $\pi(x) = \text{Li}(x) + \mathcal{O}(x^{1/2})$. This would imply

$$\frac{\pi(x)}{\text{Li}(x)} = 1 + \mathcal{O}(x^{-1/2} \log x) = 1 + \mathfrak{o}(1),$$

which gives the prime number theorem. Thus Hardy's comment, [10], page 352, that Riemann does not even state the prime number theorem, is not strictly accurate. On the other hand, Riemann's assertion about the order of the error is much stronger than what is required for the prime number theorem.

The actual expression obtained by Riemann for $\pi(x)$ is

$$\begin{aligned} \pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n}) \\ J(x) &= \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \end{aligned}$$

where ρ runs over the complex roots of the zeta function and μ is the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^m & \text{if } n \text{ is a product of } m \text{ distinct primes} \\ 0 & \text{in all other cases.} \end{cases}$$

Note that $\mu(n) = 0$ if n is not square-free.

The first sum in Riemann's expression here is actually finite for each x since

$$J(x) = \sum \frac{1}{n} \pi(x^{1/n})$$

is 0 for $x < 2$. The complete proof of Riemann's formula (in a different form) was given by von Mangoldt ([21]) in 1895.

Riemann's formula implies

$$\pi(x) - \sum_{n=1}^N \frac{\mu(n)}{n} \operatorname{Li}(x^{1/n}) = \sum_{n=1}^N \sum_{\rho} \operatorname{Li}(x^{\rho/n}) + \text{"other terms"}$$

where the omitted terms are not particularly significant. The terms in the double sum are Riemann's "periodic" terms. Individually they are quite large, but there must be a large amount of cancellation to account for the fact that

$$\sum_{n=1}^N \frac{\mu(n)}{n} \operatorname{Li}(x^{1/n})$$

gives a very close estimate of $\pi(x)$ if we take N large enough that the last term is about 1. This is the approximation to $\pi(x)$ intended by Ramanujan in his second letter to Hardy (in 1913, see [2], page 53) where he estimates $\pi(x)$ by

$$\pi(x) \approx \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{Li}(x^{1/n}). \quad (2)$$

(The series does not converge and while it is an asymptotic series, $\pi(x)$ is not an asymptotic sum of the series.)

Towards the end of his 1859 paper [14] Riemann asserts that $\pi(x) < \operatorname{Li}(x)$. This inequality is known to be true for all $x \leq 10^8$ but was proved false in general in 1914 by Littlewood, [12]. Littlewood showed that $\pi(x) - \operatorname{Li}(x)$ changes sign infinitely often. Sign changes are known to exist above $x = 10^{370}$ and may occur as soon as $x = 10^{20}$. However, a sign change has never been observed.

8 Hadamard and de la Vallée Poussin

In 1896 the prime number theorem was finally proved by Jacques Hadamard [8] and also by Charles-Jean de la Vallée Poussin [5]. The first part of the proof is to show that $\zeta(s) \neq 0$ if $\Re s = 1$. We will do this later.

As a general principle, finding zero-free regions for the zeta function in the critical strip leads to better estimates of the error in the $\pi(x) \sim \text{Li}(x)$. Thus, for example (see [11]), one has:

Theorem 8.1. *Let $\frac{1}{2} \leq \alpha < 1$. Then*

$$\psi(x) = x + \mathcal{O}(x^\alpha (\log x)^2)$$

if and only if

$$\zeta(s) \neq 0 \text{ for } \Re s > \alpha.$$

version 20020514... To be continued!

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