WEIERSTRASS VORBEREITUNGSSATZ

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For the next few lectures we will investigate some consequences of the Cauchy theorem. The topics we will consider are all closely related to the Weierstrass Vorbereitungssatz (Preparation Theorem). For simplicity, and because it's all we need, we restrict our discussion of the Preparation Theorem to the case of two variables. The general case is important for the theory of several complex variables because it forms the basis for arguments using induction on the number of variables.

Some good references for the theory of several complex variables are Bers [1], Gunning and Rossi [3] and Hörmander [4]. A perspective of the differences between one and several complex variables may be gained from section 0.3 *Comparison of* $\mathbb{C}^1$ *and* $\mathbb{C}^n$ *by way of 10 classical problems* in Krantz [5]. A nice treatment of the Preparation Theorem and a related theorem, the Division Theorem, is given in Whitney [10] and Gunning and Rossi *loc. cit.* A quantitative version is given in Hörmander *loc. cit.*

The Preparation Theorem was published in Weierstrass [8] (see also [9]). In a footnote to this paper Weierstrass says *"Diesen Satz habe ich seit dem Jahre 1860 wiederholt in meinen Universitäts–Vorlesungen vorgetragen."* (Since the year 1860 I have repeatedly demonstrated this theorem in my university lectures.)

I was somewhat surprised to find that our library has a copy of the 108 year old book [8]. The library acquired it 42 years ago. In that time, till now, it has been checked out only once – 22 years ago. The library employee who checked it out for me was not even born the last time it was checked out.

There are a large number of quotes, historical remarks and some essays concerning Abel, Cauchy, Eisenstein, Euler, Riemann and Weierstrass in the beautiful complex variable textbook of Reinhold Remmert [6]. Particularly enjoyable is the description of a joke played by Weierstrass's students in Berlin. Weierstrass was a wine connoisseur and had a Westphalian accent. The students, making fun of his accent, claimed to have heard him say *"Ein chutes Chlas Burchunder trink ich chanz chern."* The hard 'ch' sounds in this tongue challenger should of course be 'g' sounds. Remmert's book is highly recommended both as a textbook and for sheer enjoyment.

While we will need to make use of the concept of analytic functions of several variables, we don't need to know anything about them. Indeed we use nothing but the following definition. Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}$. A function

$$F : \Omega_1 \times \Omega_2 \to \mathbb{C}$$

is said to be *analytic* if it is continuous and in addition is analytic in each variable separately. A remarkable theorem of Hartogs shows that the continuity hypothesis is superfluous (see theorem 2.2.8 in [4]). Moreover one can show by using Cauchy's theorem that F is C^∞ and each of its partial derivatives is analytic.

Counting Zeros. Argument Principle

The Principle of the Argument is handy for counting roots and for proving analyticity of symmetric functions of roots. For the present we consider only analytic functions, though the same proof gives an analogous result for meromorphic functions. See our text [2], chapter 5, section 3.

Theorem 1. Argument Principle. Version 1. *Let Ω be an open subset of $\mathbb{C}$, let u and f be analytic functions in Ω and let $\alpha_1, \alpha_2, \dots, \alpha_m$ be the roots of f in Ω repeated according to multiplicity.*

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If γ is a closed rectifiable curve in $\Omega \sim \{\alpha_1, \ldots, \alpha_m\}$ and $\gamma \simeq 0$ in Ω then

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma u(z)\, \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} \nu(\gamma, \alpha_k)\, u(\alpha_k).$$

Note we are assuming that f has only finitely many roots in Ω. In practice this is not a restriction since the roots of an analytic function are isolated and we can usually shrink Ω. One might also note that $\nu(\gamma, \alpha) \neq 0$ for only finitely many roots α of f and modify the statement and proof accordingly. See [2], chapter 4, section 7, exercise 2, for part of what's needed.

Why is this theorem referred to as the Argument Principle? Recall we proved a theorem to compute the contour integral along an analytic image of a rectifiable curve — in the present context it says:

$$\nu(f \circ \gamma, 0) = \int_{f\circ\gamma} \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Thus the Argument Principle really gives an expression for the change in the argument of f along γ in terms of the winding numbers of γ about the zeros of f (at least in the case $u \equiv 1$).

Proof. By m applications of Taylor's theorem we have

$$f(z) = (z - \alpha_a)\, \ldots\, (z - \alpha_m)\, g(z)$$

where g is analytic in Ω and $g(z) \neq 0$ for each $z \in \Omega$. Thus

$$u(z)\, \frac{f'(z)}{f(z)} = \frac{u(z)}{z - \alpha_1} + \cdots + \frac{u(z)}{z - \alpha_m} + u(z)\, \frac{g'(z)}{g(z)}.$$

The last term is analytic in Ω and so its integral over γ is 0 by Cauchy's theorem. Now the proof is completed by applying the Cauchy integral formula. □

Corollary 2. *Let Ω be an open subset of $\mathbb{C}$, let f be an analytic function in Ω and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the roots of f in Ω repeated according to multiplicity. If γ is a closed rectifiable curve in $\Omega \sim \{\alpha_1, \ldots, \alpha_m\}$ and $\gamma \simeq 0$ in Ω then*

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma z^n\, \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} \nu(\gamma, \alpha_k)\, \alpha_k^n$$

for each integer $n \geq 0$.

If γ has winding number 1 about each of the roots and we take $n = 0$ the corollary yields a formula for the number of roots in Ω. If there is only one root and we take $n = 1$ the corollary yields an integral formula for the root.

The first observation suffices to prove the Open Mapping Theorem — see our text [2], chapter 4, section 7. Both observations are used in [2] to prove the Inverse Function Theorem — see chapter 5, section 3. The proof of the Implicit Function Theorem below uses the same reasoning.

Implicit Function Theorem

Theorem 3. Implicit Function Theorem. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ and let $F : \Omega_1 \times \Omega_2 \to \mathbb{C}$ be analytic. Let $z_0 \in \Omega_1$, let $w_0 \in \Omega_2$ and suppose*

$$\begin{aligned} F(z_0, w_0) &= 0 \\ \frac{\partial F}{\partial w}(z_0, w_0) &\neq 0. \end{aligned}$$

Then there exists $r_1 > 0$ and $r_2 > 0$ such that $\overline{D(z_0, r_1)} \subseteq \Omega_1$, $\overline{D(w_0, r_2)} \subseteq \Omega_2$ and for each $z \in D(z_0, r_1)$ the equation

$$F(z, w) = 0$$

has precisely <u>one</u> solution $w = h(z) \in D(w_0, r_2)$. Moreover the function

$$h : D(z_0, r_1) \to D(w_0, r_2)$$

is analytic and satisfies $h(z_0) = w_0$.

Informally the theorem says we have a unique analytic solution h of

$$\begin{aligned} F(z, h(z)) &= 0 \\ h(z_0) &= w_0 \end{aligned}$$

sufficiently near (z_0, w_0).

Proof. Since $F(z_0, w_0) = 0$ and $\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$ the function $w \hookrightarrow F(z_0, w)$ has a simple root at w_0. Hence there exists $r_2 > 0$ such that $\overline{D(w_0, r_2)} \subseteq \Omega_2$ and $F(z_0, w) \neq 0$ if $0 < |\, w - w_0 \,| \leq r_2$ (roots are isolated). Thus

$$F(z, w) \neq 0 \text{ if } (z, w) \in \{z_0\} \times \partial D(w_0, r_2).$$

Since F is continuous and $\partial D(w_0, r_2)$ is compact there exits $r_1 > 0$ such that $\overline{D(z_0, r_1)} \subseteq \Omega_1$ and

$$F(z, w) \neq 0 \text{ if } (z, w) \in \overline{D(z_0, r_1)} \times \partial D(w_0, r_2).$$

Let $\gamma(t) = w_0 + r_2\, \mathrm{e}^{\mathrm{i}t}, \quad 0 \leq t \leq 2\pi$ and let

$$k(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)}\, dw.$$

Now k is continuous, indeed analytic, on $D(z_0, r_1)$. On the other hand, for each fixed z, $k(z)$ is the number of roots of $w \hookrightarrow F(z, w)$ in $D(z_0, r_2)$ by the Argument Principle. Thus $k(z) \in \mathbb{Z}$ and so by continuity is constant. By hypothesis $k(z_0) = 1$. It follows that $k(z) = 1$ for each $z \in D(z_0, r_1)$, that is, for each $z \in D(z_0, r_1)$ there is a unique $w = h(z) \in D(w_0, r_2)$ such that $F(z, h(z)) = 0$. By the Argument Principle (with $n = 1$) we have

$$h(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma w\, \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)}\, dw$$

for each $z \in D(z_0, r_1)$. It follows that h is analytic in $D(z_0, r_1)$. ☐

If Ω is an open subset of $\mathbb{C}^2$, F is analytic in Ω, $(z_0, w_0) \in \Omega$, $r_1 > 0$, $r_2 > 0$, $\overline{D(z_0, r_1)} \times \overline{D(w_0, r_2)} \subseteq \Omega$ and F has no zeros in $\overline{D(z_0, r_1)} \times \partial D(w_0, r_2)$ then we refer to $\overline{D(z_0, r_1)} \times \partial D(w_0, r_2)$ as a *security box* at (z_0, w_0) for the zeros of F. This terminology is a bit misleading since a security box is really a three dimensional torus. The security box is handy because it allows us to use Cauchy's theorem and formula relative to w and to treat z as a parameter.

In the notation of the proof of the Implicit Function Theorem above, by Taylor's theorem, we have

$$F(z, w) = (w - h(z))\; G(z, w)$$

for $z \in D(z_0, r_1)$ and $w \in \Omega_2$ where $w \hookrightarrow G(z, w)$ is analytic in Ω_2 and $G(z, w) \neq 0$ for $w \in \overline{D(w_0, r_2)}$. The Vorbereitungssatz is of this form, but with the $w - h(z)$ replaced by a general monic polynomial.

Corollary 4. Inverse Function Theorem. Version 1. *Let Ω be an open subset of $\mathbb{C}$, $w_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ analytic. Assume $f'(w_0) \neq 0$. Then there is a neighborhood ω_2 of w_0 in Ω mapped one–to–one onto an open neighborhood ω_1 of $f(w_0)$. Moreover, the inverse map $f^{-1} : \omega_1 \to \omega_2$ is analytic.*

Proof. Let $\Omega_2 = \Omega$, $\Omega_1 = \mathbb{C}$, $z_0 = f(w_0)$ and $F(z, w) = f(w) - z$. Since $f'(w_0) \neq 0$ the implicit function theorem implies we have positive numbers r_1 and r_2 so that $\overline{D(w_0, r_2)} \subseteq \Omega$ and an analytic function $h : D(z_0, r_1) \to D(w_0, r_2)$ such that $F(z, h(z)) = 0$, that is, $f(h(z)) = z$, for each $z \in D(z_0, r_1)$. Moreover, if $z \in D(z_0, r_1)$ then $h(z)$ is the unique solution in $D(w_0, r_2)$ to the equation $F(z, w) = 0$. That is, if $z \in D(z_0, r_1)$, $\;w \in D(w_0, r_2)$ and $f(w) = z$ then $w = h(z)$. Now let $\omega_1 = D(z_0, r_1)$ and let $\omega_2 = h(\omega_1)$. By continuity of f we may choose $\delta > 0$ so that $f(D(w_0, \delta)) \subseteq D(z_0, r_1) = \omega_1$. If $w \in D(w_0, \delta)$ then $z = f(w) \in D(z_0, r_1)$ implies $w = h(z)$. Thus $D(w_0, \delta) \subseteq \omega_2$, that is, ω_2 is a neighborhood of w_0. ☐

After we prove the Vorbereitungssatz we will be able to obtain a much stronger version of the Inverse Function Theorem.

Vorbereitungssatz

Theorem 5. Vorbereitungssatz. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ and let $F : \Omega_1 \times \Omega_2 \to \mathbb{C}$ be an analytic function. Let $z_0 \in \Omega_1$, $w_0 \in \Omega_2$ and assume that $w \hookrightarrow F(z_0, w)$ is not identically zero in the component of w_0 in Ω_2. Choose $r_2 > 0$ and $r_1 > 0$ such that $\overline{D(z_0, r_1)} \subseteq \Omega_1$, $\overline{D(w_0, r_2)} \subseteq \Omega_2$ and so that $F(z_0, w) \neq 0$ on $\partial D(w_0, r_2)$ and let m be the number of roots of $w \hookrightarrow F(z_0, w)$ in $D(w_0, r_2)$, counted according to multiplicity. If necessary we may decrease $r_1 > 0$ such that*

$$F(z, w) \neq 0 \quad \text{for } z \in \overline{D(z_0, r_1)}, \quad w \in \partial D(w_0, r_2).$$

In this case, for each $z \in \overline{D(z_0, r_1)}$ the function $w \hookrightarrow F(z, w)$ has precisely m roots in $D(w_0, r_2)$ and there exist functions $A_1, \cdots, A_m$ analytic in $D(z_0, r_1)$ such that

$$F(z, w) = \left(w^m + A_1(z)\, w^{m-1} + \cdots + A_m(z)\right)\, G(z, w)$$

for each $z \in D(z_0, r_1)$ and each $w \in \Omega_2$. Here G is analytic in $D(z_0, r_1) \times \Omega_2$ and

$$G(z, w) \neq 0 \quad (z, w) \in D(z_0, r_1) \times \overline{D(w_0, r_2)}.$$

The functions $A_1, \cdots, A_m$ and G are uniquely determined by the above properties.

The function-theoretic proof we are about to give occurs, for example, in [5] and [3]. Krantz [5] attributes it to Siegel, but he doesn't give a reference.

Proof. The first part of the statement concerns the existence of a security box for the zeros of F. Since $w \hookrightarrow F(z_0, w)$ is not identically zero, its roots are isolated. Thus we can choose $r_2 > 0$ with the desired properties. Then the existence of $r_1 > 0$ follows from the continuity of F and the compactness of $\partial D(w_0, r_2)$. Let $\gamma(t) = w_0 + r_2 \mathrm{e}^{\mathrm{i}t}$, $0 \leq t \leq 2\pi$ and let

$$m(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)}\, dw.$$

Then $m(z_0) = m$ and $m(z)$ is analytic and integer valued, and so constant in $D(z_0, r_1)$. Thus for each $z \in D(z_0, r_1)$ the function $w \hookrightarrow F(z, w)$ has m zeros, say $h_1(z), \cdots, h_m(z)$, in $D(w_0, r_2)$ (counted according to multiplicity). The functions $h_1, \cdots, h_m$ can be pretty wild, since we have said nothing about how to label the zeros, but by the Argument Principle we have

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma w^k\, \frac{\frac{\partial F}{\partial w}(z, w)}{F(z, w)}\, dw = \sum_{j=1}^{m} \left(h_j(z)\right)^k$$

for each $k \in \mathbb{Z}$, $k \geq 0$. Thus the power–sum $\sum_{j=1}^m h_j^k$ is analytic in $D(z_0, r_1)$. Now by the classical theory of equations (Newton's formulae), see Uspensky [7], the elementary symmetric functions in indeterminates $X_1, \cdots, X_m$ are polynomials with rational coefficients in the power sums $Y_k = \sum_{j=1}^m X_j^k$. It follows if we define $A_1, \cdots, A_m$ by

$$w^m + A_1(z)\, w^{m-1} + \cdots + A_m(z) = (w - h_1(z))\ (w - h_2(z)) \cdots (w - h_m(z))$$

then $A_1, \cdots, A_m$ are analytic in $D(z_0, r_1)$ (since they are polynomials in the power sums). By Taylor's theorem (applied repeatedly) if $z \in D(z_0, r_1)$ then

$$F(z, w) = (w - h_1(z)) \cdots (w - h_m(z))\ G(z, w)$$

where $w \hookrightarrow G(z, w)$ is analytic in Ω_2 for each $z \in D(z_0, r_1)$ and where $G(z, w) \neq 0$ if $w \in \overline{D(w_0, r_2)}$. The polynomial

$$P(z, w) = (w - h_1(z)) \cdots (w - h_m(z))$$

for each $z \in D(z_0, r_1)$ has all its roots in $D(w_0, r_2)$. Thus $G = F/P$ is analytic in $D(z_0, r_1) \sim U$ where U is an open subset of Ω_2 which contains $\Omega_2 \sim D(w_0, r_2)$. (We chose $r_1 > 0$ so the zeros of $F(z, w)$ are bounded away from $\partial D(w_0, r_2)$ for $z \in \overline{D(z_0, r_1)}$.) On the other hand if $w \in D(w_0, r_2)$ then

$$G(z, w) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{G(z, u)}{u - w}\, du = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{F(z, u)}{P(z, u)}\, \frac{du}{u - w}.$$

Thus G is also analytic in $D(z_0, r_1) \times D(w_0, r_2)$. For uniqueness note $F(z, w) = P(z, w)\, G(z, w)$ determines P since P is monic of degree m in w and $G \neq 0$ in $D(z_0, r_1) \times \overline{D(w_0, r_2)}$. ∎

Let Ω be an open set in $\mathbb{C}$. We denote the ring of functions analytic in Ω by $\mathcal{O}(\Omega)$. A monic polynomial of degree m in $\mathcal{O}(\Omega)[w]$ is then a polynomial of the form

$$P(z,w) = w^m + A_1(z)\, w^{m-1} + \cdots + A_m(z)$$

where $A_1, \cdots, A_m$ are analytic in Ω. If $z_0 \in \Omega$ and $w \in \mathbb{C}$ then P is said to be a *Weierstrass polynomial* at (z_0, w_0) if $P(z_0, w) = (w - w_0)^m$, that is, if

$$P(z,w) = (w - w_0)^m + B_1(z)\ (w-w_0)^{m-1} + \cdots + B_m(z)$$

where $B_1, \cdots, B_m$ are analytic in Ω and

$$B_1(z_0) = B_2(z_0) = \cdots = B_m(z_0) = 0.$$

Corollary 6. Strong Implicit Function Theorem. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ and let $F : \Omega_1 \times \Omega_2 \to \mathbb{C}$ be analytic. Let $z_0 \in \Omega_1$, let $w_0 \in \Omega_2$ and suppose*

$$\begin{aligned} \frac{\partial^k F}{\partial w^k}(z_0, w_0) &= 0, \quad 0 \le k \le m-1 \\ \frac{\partial^m F}{\partial w^m}(z_0, w_0) &\neq 0. \end{aligned}$$

Then there exists $r_1 > 0$ and $r_2 > 0$ such that $\overline{D(z_0, r_1)} \subseteq \Omega_1$, $\overline{D(w_0, r_2)} \subseteq \Omega_2$ and for each $z \in D(z_0, r_1)$ the equation

$$F(z,w) = 0$$

has precisely m solutions $w = h_j(z)$ in $D(w_0, r_2)$ (counted according to multiplicity). Moreover the functions

$$h_j : D(z_0, r_1) \to D(w_0, r_2)$$

satisfy $h_j(z_0) = w_0$ and any symmetric polynomial in $h_1, \cdots, h_m$ is analytic in $D(z_0, r_1)$.

Note the assertion is that there are precisely m solutions in $D(w_0, r_2)$ for each $z \in D(z_0, r_1)$. There may be additional solutions outside of $D(w_0, r_2)$, but we don't care about them.

Proof. Choose $r_2 > 0$ and $r_1 > 0$ as usual so that $F \neq 0$ in $\{z_0\} \times \left(\overline{D(w_0, r_2)} \sim \{w_0\}\right)$ and $F \neq 0$ on $\overline{D(z_0, r_1)} \times \partial D(w_0, r_2)$. By the Preparation Theorem we have $P \in \mathcal{O}(D(z_0, r_1)[w]$ monic of degree m such that $w \hookrightarrow P(z,w)$ has all of its roots $h_1(z), \cdots, h_m(z)$ in $D(w_0, r_2)$ for each $z \in D(z_0, r_1)$ and $F = P\, G$, $G \neq 0$. These roots have all the properties claimed. □

Of course we would really like to say that the roots $h_j(z)$ are analytic functions of z, but that can't be true without specifying how to label the roots. We can avoid the problem of labelling the roots by viewing $(h_1(z), \cdots, h_m(z))$ as a function of z taking its values in a symmetric product of m copies of $\mathbb{C}$. In this context we can discuss continuity, analyticity, etc. See Whitney [10].

We finish this section with a few results we will not need in the sequel.

Note that for the polynomial P in the proof above we must have w_0 is an m–fold zero of $w \hookrightarrow P(z_0, w)$ and therefore $P(z_0, w) = (w - w_0)^m$, that is, P is a Weierstrass polynomial. One of the nice features of Weierstrass polynomials is that we can divide by them.

Theorem 7. Weierstrass Division Theorem *Let F be analytic in a neighborhood of (z_0, w_0) and let P be a Weierstrass polynomial of degree m at (z_0, w_0). Then there exists an open neighborhood U of z_0 and $R \in \mathcal{O}(U)[w]$ with degree $< m$ such that*

$$F = Q\, P + R$$

where Q is analytic in a neighborhood of (z_0, w_0). Moreover Q and R are unique.

Proof. First we note we may choose a security box for the roots of P contained in the domain where F is analytic. This is where we use that P is a Weierstrass polynomial. Explicitly we choose r_1 and r_2 so that $P \neq 0$ in each of the sets $\{z_0\} \times \left(\overline{D(w_0, r_2)} \sim \{w_0\}\right)$ and and $\overline{D(z_0, r_1)} \times \partial D(w_0, r_2)$ and small enough that $\overline{D(z_0, r_1)} \times \overline{D(w_0, r_2)}$ is contained in the domain where F is analytic. We can do this because $w \hookrightarrow P(z_0, w)$ has an m-fold root at w_0 and so we can choose r_2 as small as we please (we can always make r_1 smaller without restriction). Now define

$$Q(z,w) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(z,u)}{P(z,u)\,(u-w)}\,du$$

where $\gamma(t) = w_0 + r_2 \mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$. The Q is analytic in $D(z_0, r_1) \times D(w_0, r_2)$. Now we define R by

$$R = F - Q\,P.$$

By Cauchy's theorem applied to F we obtain

$$R(z,w) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(z,u)}{P(z,u)} \left[\frac{P(z,u) - P(z,w)}{u-w}\right] du.$$

Now the expression in the brackets is a polynomial in $u - w$ of degree $\le m-1$ with coefficients which are analytic functions of z because $u^k - w^k$ is divisible by $u - w$ if $k \ge 1$. Thus R is a polynomial of degree at most $m - 1$. For uniqueness note if we had two different remainders R_1 and R_2 then the difference $R_1 - R_2$ is divisible by P and so must have m roots, etc. □

Note the corollary is false if P is not a Weierstrass polynomial. If P is just monic we need to require that the domain of analyticity of F is large enough to contain a security box at (z_0, w_0) for the roots of P.

Note also the clever idea in the proof — we do the division away from the zeros and then use the Cauchy formula to take care of the zeros. This technique is useful in other contexts. A completely different proof, using successive approximation and careful estimates, is given in Hörmander [4].

The uniqueness part has an interesting consequence.

Corollary 8. *Suppose F and G are analytic in a neighborhood of (z_0, w_0), P is a Weierstrass polynomial of degree m at (z_0, w_0) and $F = G\,P$. If F is a polynomial in w then so is G.*

Proof. Since P is a monic polynomial we may apply the usual Euclidean algorithm to deduce

$$F = Q\,P + R$$

where R and Q are polynomials in w with coefficients which are analytic in a neighborhood of z_0 and R has degree at most $m - 1$. Then by uniqueness in the Weierstrass Division Theorem we must have $G = Q$ and $R = 0$. □

The corollary is false if P is not a Weierstrass polynomial. For example take $z_0 = 0$, $w_0 = 0$, $F = 1$, $P = w + 1$ and $G = 1/(w + 1)$.

Open Mapping Principle

Recall if X and Y are topological spaces and $f : X \to Y$ then f is said to be an *open mapping* if $f(U)$ is open in Y for each open set U in X.

Theorem 9. Open Mapping Theorem. Version 1. *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be an analytic function which is not constant on any component of Ω. Then f is an open mapping.*

Proof. Let $w_0 \in \Omega$ and let $z_0 = f(w_0)$. Since f is not constant in the component of Ω which contains w_0 there exists an integer $m \ge 1$ such that

$$\begin{aligned} f^{(k)}(w_0) &= 0, \quad 1 \le k \le m-1 \\ f^{(m)}(w_0) &\ne 0. \end{aligned}$$

Now let $F : \mathbb{C} \times \Omega \to \mathbb{C}$ be defined by $F(z,w) = f(w) - z$. Then

$$\begin{aligned} \frac{\partial^k F}{\partial w^k}(z_0, w_0) &= 0, 0 \le k \le m-1 \\ \frac{\partial^m F}{\partial w^m}(z_0, w_0) &\ne 0. \end{aligned}$$

By the Strong Implicit Function Theorem we have $F = P\,G$ in $D(z_0,\delta) \times D(w_0,\epsilon)$ for some $\delta, \epsilon > 0$ where

$$P(z,w) = (w - w_0)^m + A_1(z)\ (w - w_0)^{m-1} + \cdots\ A_m(z)$$

and where $A_1, \cdots, A_m$ are analytic in $D(z_0,\delta)$, G is analytic and nonzero in $D(z_0,\delta) \times D(w_0,\epsilon)$, $A_j(z_0) = 0,\ \ j = 1, \cdots, m$ and for each $z \in D(z_0,\delta)$ all m roots of $w \hookrightarrow P(z,w)$ lie in $D(w_0,\epsilon)$. Since $m \geq 1$ we see for each $z \in D(z_0,\delta)$ there is $w \in D(w_0,\epsilon)$ such that $P(z,w) = 0$, that is, $f(w) = z$. We have shown

$$D(z_0,\delta) \subseteq f(D(w_0,\epsilon))$$

which implies that z_0 is an interior point of the image of f. ☐

Exercise 1. *An analytic function with constant real part, or constant imaginary part, or constant modulus, is constant.*

Exercise 2. *Show that the Open Mapping Theorem implies the Maximum Modulus Theorem.*

The Open Mapping Theorem doesn't use the full strength of The Weierstrass Preparation Theorem. Here is a stronger result.

Corollary 10. Open Mapping Theorem. Version 2. *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be an analytic function which is not constant on any component of Ω. Let $w_0 \in \Omega$ and let $z_0 = f(w_0)$. Then there exists an integer $m \geq 1$ such that*

$$\begin{aligned} f^{(k)}(w_0) &= 0, \quad 1 \leq k \leq m-1 \\ f^{(m)}(w_0) &\neq 0. \end{aligned}$$

Moreover there exist numbers $\epsilon, \delta > 0$ such that each point in $D(z_0,\delta) \sim \{z_0\}$ is the image of precisely m distinct points in $D(w_0,\delta) \sim \{w_0\}$.

Actually one can show even more — that there exists an analytic function ϕ such that the m solutions of $f(w) = z$ are given by

$$w = \phi((z - z_0)^{1/m})$$

where we use each determination of $(z - z_0)^{1/m}$. We will do this later and as a consequence obtain a direct proof of version 2 of the Open Mapping Theorem.

Proof. Since f is not constant on any component of Ω we know that f' is not identically zero on any component of Ω. Since f' is analytic it must then have isolated zeros. We don't know whether or not $f'(w_0)$ but we can assert that, making the ϵ in the proof above smaller if necessary, we must have $f'(w) \neq 0$ for $w \in D(w_0,\epsilon) \sim \{w_0\}$. (In the proof of the Implicit Function Theorem we choose $\epsilon = r_2$ first and then find $\delta = r_1$.) It follows if $z \neq z_0$ that $w \hookrightarrow f(w) - z$ has only simple zeros in $D(w_0,\epsilon)$. Since there must be m zeros, the equation $f(w) = z$ must have m distinct solutions. ☐

Inverse Function Theorem

Theorem 11. Inverse Function Theorem. Version 2. *Let Ω be open in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be analytic and one–to–one. Then $\Omega_1 = f(\Omega)$ is open and $f^{-1} : \Omega_1 \to \Omega$ is analytic. In particular $f'(z) \neq 0$ for each $z \in \Omega$.*

Proof. By the Open Mapping Theorem Ω_1 is open. Let $w_1 \in \Omega$ and let $z_1 = f(w_1)$. Since f is one–to–one the function $w \hookrightarrow f(w) - z_1$ must have a simple root at w_1, that is, $m = 1$ where m is the integer which occurs in the second version of the Open Mapping Theorem. It follows that $f'(w_1) \neq 0$. Now version 1 of the Inverse Function Theorem implies that f^{-1} is analytic. ☐

Proof. **(Alternate)** Let $z_0 \in \Omega_1$ and let $z_0 = f(w_0)$. Since f is one–to–one the function $w \hookrightarrow f(w) - z_0$ has simple roots. Thus the degree of the Weierstrass polynomial which occurs in the Generalized Implicit Function Theorem must be 1, that is,

$$f(w) - z = [(w - w_0) + A_1(z)]\ G(z,w)$$

for $z \in D(z_0, \delta)$ and $w \in \Omega$ where $G(z, w) \neq 0$ if $z \in \overline{D(z_0, \delta)}$ and $w \in \overline{D(w_0, \epsilon)}$. Now if $z \in D(z_0, \delta)$ there is $w \in D(w_0, \epsilon)$ such that $f(w) = z$ and it's given by $w = w_0 + A_1(z)$. Since f is one–to–one this w is the only solution in Ω. We have shown

$$f^{-1}(z) = w_0 + A_1(z) \text{ if } z \in D(z_0, \delta).$$

Thus f^{-1} is analytic. □

Let $f : \Omega \to \mathbb{C}$ be a one–to–one analytic function. Let $w_0 \in \Omega$ and choose $r > 0$ such that $\overline{D(w_0, r)} \subseteq \Omega$. Let $\gamma(t) = w_0 + r\,\mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$. If $w \in D(w_0, r)$ and $f(w) = z$ then w is the unique root of $u \hookrightarrow f(u) - z$ and therefore by the Argument Principle we have

$$w = \frac{1}{2\pi\mathrm{i}} \int_\gamma u \frac{f'(u)}{f(u) - z}\, du$$

that is

$$f^{-1}(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma u \frac{f'(u)}{f(u) - z}\, du$$

for each $z \in f(D(w_0, r))$. If $z_0 = f(w_0)$ then from the fact that f^{-1} is analytic in $f(\Omega)$ we have

$$f^{-1}(z) = \sum_{n=0}^{\infty} b_m\ (z - z_0)^n \text{ for } |\, z - z_0\, | < \delta \text{ if } D(z_0, \delta) \subseteq f(\Omega).$$

The coefficients $b_n = \frac{1}{n!}\left(f^{-1}\right)^{(n)}(z_0)$ we can compute by differentiating the integral. We obtain

$$b_n = \frac{1}{2\pi\mathrm{i}} \int_\gamma w \frac{f'(w)}{\left(f(w) - z_0\right)^{n+1}}\, dw.$$

These integrals, unlike the one for f^{-1} above, can sometimes be computed by means of the Residue Theorem.

References

1. Lipman Bers, *Introduction to several complex variables. Lecture notes 1962–1963*, Courant Institute of Mathematical Sciences. New York Univ., New York, 1964.
2. John B. Conway, *Functions of one complex variable*, 2 ed., Graduate Texts in Mathematics, vol. 11, Springer–Verlag, New York · Berlin · Heidelberg · London · Paris · Tokyo · Hong Kong · Barcelona, 1978.
3. Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice–Hall, Inc., Englewood Cliffs, N. J., 1965.
4. Lars Hörmander, *An introduction to complex analysis in several variables*, D. van Nostrand Co., Inc., Princeton, New Jersey · Toronto · New York · London, 1966.
5. Steven G. Krantz, *Function theory of several complex variables*, John Wiley & Sons, New York · Chichester · Brisbane · Toronto · Singapore, 1982.
6. Reinhold Remmert, *Theory of complex functions*, Graduate Texts in Mathematics, *Readings in Mathematics*, vol. 122, Springer–Verlag, New York · Berlin · Heidelberg · London · Paris · Tokyo · Hong Kong, 1991 (Translated from 2nd German edition of *Funktionentheorie I*, Grundwissen Mathematik, vol. 5, 1989).
7. J. V. Uspensky, *Theory of equations*, McGraw–Hill Book Co., New York · Toronto · London, 1948.
8. Karl Weierstrass, *Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze*, Abhandlungen aus der Functionenlehre von Karl Weierstrass, Verlag von Julius Springer, Berlin, 1886, pp. 105–164.
9. ———, *Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze*, Mathematische Werke von Karl Weierstrass, vol. 2, Mayer & Müller, Berlin, 2 ed., 1895, pp. 135–188.
10. Hassler Whitney, *Complex analytic varieties*, Addison–Wesley Publ. Co., Reading, Massachusetts · Menlo Park, California · London · Don Mills, Ontario, 1972.

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