Calculus: Sequences and Series – Mth 253

Archive – Fall 1995 Files

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This document contains two quizzes and the final exam from Mth 253 Spring 1995. The original test formatting has not been preserved.

Contents

1 Quiz 1

Problem 1. Find the limits:

Part (A):

$$\lim_{x\to 0} \frac{e^{2x} - 1}{\log(1-x)}$$

Part (B):

$$\lim_{x\to 0} \frac{2x - \sin(2x)}{2x + \sin(2x)}$$

Part (C):

$$\lim_{x\to 0} \frac{e^{x} + e^{-x} - x^2 - 2}{\sin^2(x) - x^2}$$

Problem 2. For each of the following improper integrals, indicate if the integral converges or diverges. If an integral converges, then evaluate it.

Part (A):

$$\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{1-\cos(x)}}\, dx$$

Part (B):

$$\int_{-\infty}^{\infty} \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}\, dx$$

Part (C):

$$\int_0^1 \frac{dx}{x^{3/2}}$$

Part (D):

$$\int_1^{\infty} \frac{dx}{x^{3/2}}$$

Problem 3.

Part (A): For a certain infinite series the n^{th} partial sum s_n is found to be

$$s_n = \frac{(2n-1)^2}{(3n+2)(n+3)}.$$

Find the sum of the series.

Part (B): Sum the series

$$\sum_{n=0}^{\infty} \frac{(-2)^n + 2^n}{3^n}.$$

Problem 4. For each of the following series determine if the series *converges absolutely*, *converges conditionally*, or *diverges*. Be sure to justify your conclusion carefully.

Part (A):

$$\sum_{n=1}^{\infty} (-1)^n\, n^{-1/n}$$

Part (B):

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Part (C):

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{n+2}{2n+1}\right)^n$$

Part (D):

$$\sum_{n=0}^{\infty} \frac{8^{2n}\,(n\,!)^3}{3^n\,(3n)\,!}$$

Problem 5. Test for convergence or divergence:

Part (A):

$$\sum_{n=0}^{\infty} \frac{5^n}{3^n + 4^n}$$

Part (B):

$$\sum_{n=0}^{\infty} \frac{4^n}{3^n + 5^n}$$

Problem 6. Given

$$\mathrm{e}^{-1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n\,!}$$

how many terms do we need to add in order to estimate e^{-1} with an error no bigger than 0.000003?

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2 Quiz 2

Problem 7. Find the Taylor–Maclaurin polynomial of degree 4 for the function f given by

$$f(x) = \mathrm{e}^{(x-x^2)}.$$

Hint: The Taylor–Maclaurin polynomial is the Taylor polynomial at the origin.

Problem 8. The Taylor–Maclaurin polynomial of degree 4 of a function f is known to be

$$P(x) = 3 - 2\,x + \frac{3}{5}\,x^2 + \frac{7}{24}\,x^3 - \frac{35}{144}\,x^4.$$

Part (A): Compute

$$f^{(4)}(0) =$$

Part (B): Suppose we know

$$\left| f^{(5)}(x) \right| \le 3 \quad \text{for} \quad 1 \le x \le 1.$$

Find an upper bound for $|\, f(x) - P(x) \,|$ for $|\, x \,| \le 1$.

Hint: Recall Taylor's theorem

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1}\,.$$

for some ξ between x and a.

Problem 9. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+2)^{20n}}{n^{100}}.$$

Problem 10. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^3\; x^{2n}}{(3n)!\,5^n}.$$

Problem 11.

Part (A): Simplify

$$(2-3\mathrm{i})(1+4\mathrm{i}) =$$

.

Part (B): Simplify

$$\frac{3-2\mathrm{i}}{2-5\mathrm{i}} =$$

.

Part (C): Express in *polar form*

$$12 - 5\mathrm{i} =$$

.

Part (D): Write Euler's formula.

Part (E): Compute

$$\mathrm{e}^{7\pi\mathrm{i}/6} =$$

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3 Final Exam

Problem 12. For each of the following series determine if the series *converges absolutely*, *converges conditionally* or *diverges*. Be sure to give a complete and careful explanation of your reasoning.

Part (A): $\displaystyle\sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$

Part (B): $\displaystyle\sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

Part (C): $\displaystyle\sum_{n=13}^{\infty}\frac{1}{n\,\log n}$

Part (D): $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

Part (E): $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}.$

Problem 13.

If $P_n(x)$ is the Taylor polynomial centered at a, of degree $\leq n$, for the function f, then the error in $P_n(x)$, when we view $P_n(x)$ as an approximation of $f(x)$, is

$$f(x)-P_n(x)=\frac{1}{n\,!}\int_a^x (x-t)^n\ f^{(n+1)}(t)\,dt=\frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}(x-a)^{n+1}$$

where ξ_x is some point between the numbers a and x.

Suppose f is a five times continuously differentiable function and $f(1)=2$, $f'(1)=-2$, $f''(1)=4$, $f^{(3)}(1)=-8$, and $f^{(4)}(1)=-6$.

Part (A): Find the Taylor polynomial $P_4(x)$ of f of degree ≤ 4 with center at 1.

Part (B): Suppose

$$|f^{(5)}(x)|\leq 4 \ \text{ for each } \ x\in[0,3].$$

Find an estimate for the error $|f(x)-P_4(x)|$ for $x\in[0,3]$.

Problem 14.

For each of the following power series compute the radius of convergence:

Part (A):

$$\sum_{n=7}^{\infty} \frac{(n!)^2 \; x^{2n}}{(2n)!}.$$

Part (B):

$$\sum_{n=3}^{\infty} \frac{n!\, x^{3n}}{n^{10}}.$$

Problem 15.

Part (A): If $z = 5 - 12\mathrm{i}$ compute $|\,z\,|$.

Part (B): If $u = 2 + 3\mathrm{i}$ and $v = 4 - \mathrm{i}$ simplify $\dfrac{u}{v}$.

Part (C): If $z = -3 + 3\mathrm{i}$ find the *polar form* of z.

Part (D): Compute $\mathrm{e}^{\pi \mathrm{i}}$.

Part (E): If $z = 2 + 5\mathrm{i}$ and $w = 3 - 2\mathrm{i}$ compute $z\overline{w}$ where $\overline{w}$ denotes the *conjugate* of w.

Problem 16.

Part (A): If

$$z = x^2 + 2\,xy^2 - 2\,y^2$$

then find the partial derivatives

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}$$

at the point (x, y) where $x = 1$ and $y = 2$.

Part (B): Find the tangent plane to the graph

$$z = x^2 + 2\,xy^2 - 2\,y^2$$

at the point $(1, 2, 1)$.

Problem 17.

Find the volume under the graph of the elliptic paraboloid

$$z = 2\,x^2 + 3\,y^2$$

and above the rectangle Ω in the xy–plane given by

$$\Omega = \{\, (x, y) \ \mid\ 0 \le x \le 1,\ \ 0 \le y \le 2 \,\}\,.$$

Problem 18.

Evaluate the double integral

$$\iint\limits_{\Omega} \mathrm{e}^{x^2}\, dxdy$$

where

$$\Omega = \{\, (x, y) \ \mid\ 0 \le y \le x,\ \ 0 \le x \le 2 \,\}$$

is the triangle bounded by the line $y = x$, the line $x = 2$ and the x–axis.

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