Mth 256 Final Exam | Fall 2007 | (Scantron required)

Bent Petersen `256f2007-exam.tex`

Time: 110 min. Date: Dec 5 2007. Location: § 010 in PHAR 305, § 020 in KIDD 364

- A scantron is provided with this test. Fill in your ID information on the scantron now. Do not fold, staple or tear, etc., the scantron.
- This test consists of multiple-choice problems. Fill in the answers to the problems on the scantron. Return only the scantron. You may keep the test.
- Tables of integrals and Laplace transforms are attached at the end of the test.
- Depending on your solution methods your answers may appear in a different form from the ones provided on the test. You are expected to be able to provide the appropriate manipulations to identify the correct answer.
- You may use one 8.5×11 inch note sheet prepared in advance. Note sheets may not be shared. If you do not bring a note sheet you will have to do without any help notes. You may not use any books, notebooks, additional note sheets nor note cards.
- You may use a simple scientific calculator or a modest graphics calculator on this test and you are expected to have one available. An overly elaborate calculator, laptop, handheld or notebook computer, or any device capable of extensive symbolic manipulation (other than your own brain) will not be allowed. Calculators and other equipment may not be shared.
- During the test be sure to check the board occasionally for corrections. Note $\log(x)$ means the natural logarithm of x.
- There are 15 multiple-choice problems worth 12 points each,

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Problem 1. If we use the method of undetermined coefficients to find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} - y = 4te^t$$

we should use a proposed solution of the form

A.) $(At+B)e^t$ **B.)** $(At+B)te^t$

C.) $Ate^t + B$ **D.)** $At^2e^t + B$ **E.)** None of the foregoing.

☐ *←Mark your answer here and on the scantron.* *(Problem 1).*

Problem 2. If we use the method of undetermined coefficients to find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = t\sin(t)$$

we should use a proposed solution of the form

A.) $(At+B)\sin(t)$ **B.)** $At\sin(t) + Bt\cos(t)$

C.) $(At+B)\sin(t) + (Dt+E)\cos(t)$ **D.)** $(At+B)t\sin(t) + (Dt+E)t\cos(t)$

E.) None of the foregoing.

☐ *←Mark your answer here and on the scantron.* *(Problem 2).*

Problem 3. The ordinary differential equation

$$(1-t)\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=0$$

has a fundamental solution set $y_1(t)=t$, $y_2(t)=e^t$. Find the solution $y(t)$ with initial values $y(2)=3$, $y'(2)=1$.

A.) $3e^{t-2}$ **B.)** $3t/2$

C.) $2t-e^{t-2}$ **D.)** $2t-2e^t$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 3).*

Problem 4. Use the method of variation of parameters to find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dt^2}+y=\sec^2(t).$$

A.) $t\sin(t)+\log|\cos(t)|\cos(t)$ **B.)** $-\log|\sin(t)+\tan(t)|\cos(t)$

C.) $-2+\log|\sec t+\tan t|\sin(t)$ **D.)** $-1+\log|\sec(t)+\tan(t)|\sin(t)$

E.) None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 4).*

Problem 5. Consider the ordinary differential equation

$$(1-t)\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=g(t)$$

where g is a continuous function on the interval $t>1$. A fundamental solution set for the associated homogeneous equation (that is, the complementary equation) is given by $y_1(t)=t$ and $y_2(t)=e^t$. Use the method of variation of parameters to find a solution of the form $y=ut+ve^t$ for the inhomogeneous equation. Then

A.) $u'(t)=g(t)/(1-t)^2$ **B.)** $u'(t)=-tg(t)e^{-t}/(1-t)^2$

C.) $u'(t)=g(t)/(1-t)$ **D.)** $u'(t)=-tg(t)e^{-t}/(1-t)$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 5).*

Problem 6. For a certain (dissipative) spring with a periodic forcing term the equation for the displacement is found to be

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4\cos(2t) + \sin(2t).$$

The motion consists of exponentially decaying terms and sinusoidal periodic terms (the steady state). Find the amplitude of the steady state.

A.) 1 **B.)** $\sqrt{17}$
C.) 17 **D.)** 5 **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 6).*

Problem 7. For the harmonic oscillator

$$\frac{d^2y}{dt^2} + 9y = 0$$

with initial conditions $y(0) = 4$ and $y'(0) = -3$ find the amplitude of the (sinusoidal periodic) solution.

A.) 5 **B.)** 7
C.) 17 **D.)** 19 **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 7).*

Problem 8. Compute the Laplace transform of

$$4\mathrm{e}^{t} - 3\mathrm{e}^{-t}.$$

A.) $(s+7)/(s^2-1)$ **B.)** $(7s+1)/(s^2-1)$
C.) $8s/(s^2-1)$ **D.)** $6/(s^2-1)$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 8).*

Problem 9. Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{4s^3}{s^4-1}\right\}.$$

A.) $-\cos(t) + \sin(t) + \mathrm{e}^{-t}$ **B.)** $-2\cos(t) + \mathrm{e}^{t} + \mathrm{e}^{-t}$
C.) $2\sin(t) + \mathrm{e}^{t} - \mathrm{e}^{-t}$ **D.)** $2\cos(t) + \mathrm{e}^{t} + \mathrm{e}^{-t}$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 9).*

Problem 10. Consider the initial value problem

$$\frac{d^2y}{dt^2} + 4y = \sin(2t), \qquad y(0) = 0, \;\; y'(0) = 1.$$

Find the Laplace transform $Y(s)$ of the solution $y(t)$.

A.) $1/(s^2+4)^2$ **B.)** $(s+6)/(s^2+4)^2$
C.) $(s^2+6)/(s^2+4)^2$ **D.)** $(s+6)/(s^2+4)$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 10).*

Problem 11. Consider the initial value problem

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = \exp(3t), \quad y(0) = 1, \;\; y'(0) = 0$$

Find the Laplace transform $Y(s)$ of the solution $y(t)$.

A.) $\frac{s^2-2s-2}{(s-2)(s-3)(s+3)}$ **B.)** $\frac{s^2-2s-2}{(s-2)(s-3)^2}$
C.) $\frac{s^2+2s-2}{(s-2)(s+3)^2}$ **D.)** $\frac{s^2+2s-2}{(s-2)(s-3)(s+3)}$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 11).*

Problem 12. Find the Laplace transform of

$$u\left(t - \frac{\pi}{2}\right)\sin(t)$$

where u is the unit step function (Heaviside function).

A.) $e^{-s\pi/2}\frac{s}{s^2+1}$ **B.)** $e^{-s\pi/2}\frac{1}{s^2+1}$
C.) $e^{-s\pi/2}\frac{s+1}{s^2+1}$ **D.)** $e^{-s\pi/2}\frac{s-1}{s^2+1}$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 12).*

Problem 13. A cold beer, initially at $40°$ F is brought into a warm room. A few minutes later the beer is found to be $46°$ F. The same length of time later it is a perfect $51°$ F. Use Newton's law of cooling to determine the temperature of the room.

A.) $72°$ F **B.)** $74°$ F
C.) $76°$ F **D.)** $78°$ F **E.)** $80°$ F

☐ ←*Mark your answer here and on the scantron.* *(Problem 13).*

Problem 14. Suppose a tank, containing a certain fluid, has an outlet near the bottom. If h is the head, that is, the height of the fluid surface above the outlet, then Torricelli's principle states that the outflow velocity at the outlet is equal to the free-fall velocity of a particle at height h, that is, $\sqrt{2gh}$. This principle leads to the ordinary differential equation

$$A(h)\frac{dh}{dt} = -\gamma\, a\sqrt{2gh}$$

where

$A(h)$	=	horizontal cross-sectional area of the tank at height h
a	=	cross-sectional area of the outlet
g	=	acceleration of gravity, 32.1 ft/sec^2
γ	=	Borda's factor

Note Borda's factor accounts for contraction of the (assumed smooth) outflow stream. It is about 0.60 for water so we assume $\gamma = 0.60$. Borda was a contemporary of E. Torricelli 1608–1647.

Consider a tank (open on top) with a constant horizontal cross-sectional area of 20 ft^2 and a height of 9 ft above an outlet. Suppose the outlet has a cross-sectional area of 0.0218 ft^2. If the tank is initially full of water, how long does it take (in seconds) to drain down to the level of the outlet? Choose the closest number.

A.) 1050 **B.)** 1150
C.) 1250 **D.)** 1350 **E.)** 1450

☐ ←*Mark your answer here and on the scantron.* *(Problem 14).*

Problem 15. Find the Laplace transform of the solution to the initial value problem

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = \delta(t-\pi), \quad y(0) = 0, y'(0) = 0.$$

A.) $e^{-\pi s}/(s^2+6s+10)$ **B.)** $e^{-s}/(s^2+6s+10)$
C.) $1/(s^2+6s+10)$ **D.)** $(s-\pi)/(s^2+6s+10)$ **E.)** None of the foregoing.

☐ ←*Mark your answer here and on the scantron.* *(Problem 15).*

Some Laplace exchange formulæ

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = F(s-a)$$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(r)\ \mathrm{d}r$$

$\left(\text{if } \dfrac{f(t)}{t} \text{ integrable at 0}\right)$

$$\mathcal{L}\left\{\int_0^t f(r)\ \mathrm{d}r\right\}(s) = \frac{F(s)}{s}$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\}(s) = sF(s) - f(0)$$

(if f cont. on $[0,\infty)$)

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\}(s) = s^2F(s) - sf(0) - f'(0)$$

(if $f,\ f'$ cont. on $[0,\infty)$)

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$$

($u =$ unit step function)

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$ then $\mathcal{L}\{(f*g)(t)\}(s) = F(s)G(s)$ where $f*g$ is defined by $(f*g)(t) = \int_0^t f(t-r)g(r)\ \mathrm{d}r$.

Some Laplace transforms

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

$$\mathcal{L}\left\{e^{at}\right\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\cos\omega t\}(s) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\}(s) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cos\omega t\right\}(s) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\sin\omega t\right\}(s) = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left\{e^{at}\cosh\nu t\right\}(s) = \frac{s-a}{(s-a)^2-\nu^2}$$

$$\mathcal{L}\left\{e^{at}\sinh\nu t\right\}(s) = \frac{\nu}{(s-a)^2-\nu^2}$$

$$\mathcal{L}\left\{\sqrt{t}\right\}(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\left\{t^n e^{at}\right\}(s) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s} \quad (u = \text{ unit step function})$$

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as} \quad (\delta = \text{ Dirac delta})$$

Additional Laplace transforms

If f is periodic with period $T > 0$ then $\mathcal{L}\{f(t)\} = \dfrac{\int_0^T \mathrm{e}^{-st} f(t)\ \mathrm{d}t}{1 - \mathrm{e}^{-sT}}$.

$$\mathcal{L}\left\{1 + \sum_{k=1}^{\infty} (-1)^k\, u(t-k)\right\}(s) = \frac{1}{s\,(1 + \mathrm{e}^{-s})}$$

$$\mathcal{L}\{|\sin(t)|\}(s) = \frac{\coth\left(\frac{\pi s}{2}\right)}{1 + s^2}$$

Some useful integrals

$$\int \tan(t)\ \mathrm{d}t = \log|\sec(t)|$$

$$\int \tan^2(t)\ \mathrm{d}t = \tan(t) - t$$

$$\int \tan^3(t)\ \mathrm{d}t = \frac{1}{2}\tan^2(t) - \log|\sec(t)|$$

$$\int \sec(t)\ \mathrm{d}t = \log|\tan(t) + \sec(t)|$$

$$\int t\ \sin(t)\ \mathrm{d}t = \sin(t) - t\ \cos(t)$$

$$\int t\ \cos(t)\ \mathrm{d}t = \cos(t) + t\ \sin(t)$$

Use the backs of the test pages for scratch work.

Enjoy yout Winter break!