

Applied Differential Equations – Mth 256

Archive – Fall 2000 Files

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This archive contains the tests from Mth 256 Fall. The original test instructions, headers and formatting have not been preserved. Some minor errors were corrected, and possibly some new ones were introduced.

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1 Test 1

Problem 1. (15 points if correct, -3 points if wrong). Find m such that y^m is an integrating factor for the ODE

$$-2xy + (2x^2 + y) \frac{dy}{dx} = 0.$$

- A.) -2 B.) -1
C.) 2 D.) 3 E.) None of the above.

← Letter corresponding to your answer to problem 1.

Problem 2. (30 points). Solve the initial value problem

$$\frac{dy}{dt} - t^{-3}y = 2t^{-3}, \quad y(1) = 5.$$

Problem 3. (30 points). Solve the initial value problem

$$2x^2 \frac{dy}{dx} = x^2 + y^2, \quad y(1) = 2.$$

Problem 4. (30 points). Solve the initial value problem

$$\cos(t) \frac{dy}{dt} + \sin(t)y = 3 \sin(t), \quad y(0) = 5.$$

Problem 5. (30 points). Solve the initial value problem

$$t \frac{dy}{dt} = y^2, \quad y(1) = 2.$$

Problem 6. (15 points if correct, -3 points if wrong). Find the orthogonal trajectories to the 1-parameter family of parabolas

$$2x + \alpha = y^2.$$

Here α is the parameter. (In the proposed solutions β is the parameter.)

- A.) $y = e^{-\beta x}$ B.) $y = \beta e^{-x}$
C.) $y = e^{\beta x}$ D.) $2y + \beta = x^2$ E.) None of the above.

← Letter corresponding to your answer to problem 6.

2 Test 2

Problem 7. (25 points). Solve (explicitly) the nonlinear initial value problem

$$\frac{d^2y}{dt^2} = y \frac{dy}{dt}, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}.$$

Problem 8. (50 points). Find the general solution in real form for each of the following differential equations:

(Part A)

$$\frac{d^4y}{dt^4} - 4 \frac{d^3y}{dt^3} = 0$$

(Part B)

$$\frac{d^4y}{dt^4} - 4 \frac{d^2y}{dt^2} = 0$$

(Part C)

$$\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 29y = 0$$

(Part D)

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - 2 \frac{dy}{dt} = 0$$

(Part E)

$$\frac{d^4y}{dt^4} - 4 \frac{d^3y}{dt^3} + 6 \frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + y = 0$$

Problem 9. (50 points). Find the general solution in real form for each of the following differential equations:

(Part A)

$$t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} - 10y = 0$$

(Part B)

$$t^2 \frac{d^2 y}{dt^2} + 7t \frac{dy}{dt} + 13y = 0$$

(Part C)

$$t^2 \frac{d^2 y}{dt^2} + 5t \frac{dy}{dt} + 4y = 0$$

(Part D)

$$\frac{d^2 y}{dt^2} + \left(\frac{dy}{dt} \right)^2 = 0$$

(Part E)

$$t \frac{d^4 y}{dt^4} = \frac{d^3 y}{dt^3}$$

Problem 10. (25 points). A uniform chain of length l and mass m lies at rest on a wet and slippery dock with one end hanging vertically over the edge. The initial length of the overhanging piece is ϵl where $0 < \epsilon < 1$. At a certain moment, ($t = 0$), a small disturbance causes the chain to begin to slide off the dock. If y is the length of the overhanging piece, then prior to the chain sliding off completely, the total energy E is given by

$$E = \frac{1}{2}m \left(\frac{dy}{dt} \right)^2 - \frac{y}{2}m \frac{y}{l}g$$

where g is the acceleration of gravity. From conservation of energy one obtains immediately

$$\frac{d^2 y}{dt^2} = \frac{y}{l}g \quad \text{if } 0 \leq t \leq T$$

where T is the exact time the chain slides completely off the dock. **(Part A)** Find an expression for $y(t)$ valid for $0 \leq t \leq T$. Note you will have to sort out the initial conditions from the text above. **(Part B)** Find an equation which may be used to determine T . You need not solve for T (though it is not difficult to do so).

3 Exam

Problem 11. (35 points). A 100 liter tank initially contains 50 liters of brine of concentration $\frac{1}{2}$ gram salt per liter. Brine of concentration 2 grams salt per liter begins to flow into the tank at 3 liters per minute. The well-mixed solution is drawn off from the tank at 2 liters per minute. Find the concentration of salt in the tank at the very moment of overflow of the tank.

Problem 12. (30 points). Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + y = \sec(t) \tan^2(t)$$

Problem 13. (30 points). The method of undetermined coefficients provides a form for a particular solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^t \cos(2t) + e^{-t} \sin(2t).$$

Do not solve the equation – just give the form that the method of undetermined coefficients provides. Your solution should have several undetermined coefficients in it – please label them as A,B,C,...

Problem 14. (30 points). The method of undetermined coefficients provides a form for a particular solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = t e^{-2t} + e^{2t} + e^{3t}.$$

Do not solve the equation – just give the form that the method of undetermined coefficients provides. Your solution should have several undetermined coefficients in it – please label them as A,B,C,...

Problem 15. (30 points). Find the inverse Laplace transform

$$\mathcal{L}^{-1} \left\{ \frac{2s^2 - 5}{s(s-1)(s+2)} \right\}.$$

Problem 16. (30 points). Find the inverse Laplace transform

$$\mathcal{L}^{-1} \left\{ \frac{s^3 - 2s^2 + s + 1}{s^2(s^2 + 1)} \right\}.$$

Problem 17. (30 points). Find the Laplace transform of the solution to the following initial value problem. Do not bother to find the solution of the initial value problem.

$$\begin{aligned} 4 \frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + 3y(t) &= 1 + 3 \cos(t), \\ y(0) = -2 \quad , \quad y'(0) &= 1. \end{aligned}$$

Problem 18. (35 points). Find the solution to the initial value problem.

$$t^2 \frac{d^2 y}{dt^2} - 2y = t^2, \quad y(1) = 3, \quad y'(1) = -1.$$

The following three tables were on the final exam.

Some useful integrals

$$\int \tan(t) \, dt = \log |\sec(t)|$$

$$\int \tan^2(t) \, dt = \tan(t) - t$$

$$\int \tan^3(t) \, dt = \frac{1}{2} \tan^2(t) - \log |\sec(t)|$$

$$\int \sec(t) \, dt = \log |\tan(t) + \sec(t)|$$

$$\int t \sin(t) \, dt = \sin(t) - t \cos(t)$$

$$\int t \cos(t) \, dt = \cos(t) + t \sin(t)$$

Some Laplace transforms

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\cos \omega t\}(s) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\{\sin \omega t\}(s) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\{e^{at} \cos \omega t\}(s) = \frac{s-a}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\{e^{at} \sin \omega t\}(s) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\{e^{at} \cosh \nu t\}(s) = \frac{s-a}{(s-a)^2 - \nu^2}$$

$$\mathcal{L}\{e^{at} \sinh \nu t\}(s) = \frac{\nu}{(s-a)^2 - \nu^2}$$

$$\mathcal{L}\{\sqrt{t}\}(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s} \quad (u = \text{unit step})$$

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as} \quad (\delta = \text{Dirac delta})$$

Some Laplace exchange formulæ

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(r) dr \quad \left(\text{if } \frac{f(t)}{t} \text{ integrable at } 0\right)$$

$$\mathcal{L}\left\{\int_0^t f(r) dr\right\}(s) = \frac{F(s)}{s}$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\}(s) = sF(s) - f(0) \quad (\text{if } f \text{ cont. on } [0, \infty))$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\}(s) = s^2F(s) - sf(0) - f'(0) \quad (\text{if } f, f' \text{ cont. on } [0, \infty))$$

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s) \quad (u = \text{unit step})$$

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$ then $\mathcal{L}\{(f*g)(t)\}(s) = F(s)G(s)$ where $f*g$ is defined by $(f*g)(t) = \int_0^t f(t-r)g(r) dr$.

4 Contact Information

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