Applied Differential Equations – Mth 256

Archive – Spring 1996 Files

Oct 10, 2000

This archive contains the tests from Mth 256 Spring 1996. The original test instructions, headers and formatting have not been preserved.

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1 Test 1

Problem 1. *(20 points).* A 1000 gallon tank initially contains 600 gallons of brine of concentration 1.0 oz salt per gallon. Brine of concentration 2.5 oz salt per gallon flows into the tank at 6 gallons per minute. The well-mixed solution is pumped out at the rate of 4 gallons per minute. Find the concentration of the brine solution in the tank at the very moment that the tank begins to overflow.

Problem 2. *(20 points).* Use the substitution $w = y^{-2}$ to solve the initial value problem

$$y' + y = y^3, \quad y(0) = \frac{1}{2}.$$

Problem 3. *(20 points).* The differential equation

$$\left(2x^2 + y\right)\, dx + \left(x^2 y - x\right)\, dy = 0$$

has an integrating factor of the form $\mu = x^k$. **(A)** Find the integrating factor μ. **(B)** Find the solution of the differential equation satisfying the initial condition $y(1) = 2$.

Problem 4. *(20 points).* Solve the (homogeneous) initial value problem

$$x\,\frac{dy}{dx} = y + x\,\sec\left(y/x\right), \quad y(1) = \frac{\pi}{4}.$$

Problem 5. *(20 points).* Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing linearly (for a while at least), say $A = A_0 + A_1\,t$ where A_0 and A_1 are constants, and t is time. According to Netwon's law of cooling we have

$$\frac{dT}{dt} = -k\,\left(T - A\right)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1 and k. (Do not neglect the arbitrary constant.)

2 Test 2

Problem 6. *(10 points).* Suppose $\phi(x)$ is a particular solution of the ordinary differential equation $y''+p(x)y'+q(x)y = g(x)$. Suppose that $y_1(x)$ and $y_2(x)$ are complementary solutions. Suppose moreover that $y_1(a) = 1$, $y_1'(a) = 2$, $y_2(a) = -2$, $y_2'(a) = 2$, $\phi(a) = 3$ and $\phi'(a) = -1$.

Find a solution $y(x)$ (in terms of $y_1(x)$, $y_2(x)$ and $\phi(x)$) of the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \quad y(a) = -1, \quad y'(a) = 1.$$

Problem 7. *(40 points).* For each of the following linear homogeneous ordinary differential equations find the general solution (fundamental solution) in real form:

(A). $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6\,y = 0$

(B). $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = 0$

(C). $\dfrac{d^2y}{dx^2} + 4\,\dfrac{dy}{dx} + 13\,y = 0$

(D). $\dfrac{d^2y}{dx^2} + 4\,\dfrac{dy}{dx} + 4\,y = 0$

(E). $\dfrac{d^2y}{dx^2} + 16\,y = 0$

(F). $\left(D^2+4\right)^2\ y = 0$

(G). $(D-2)^5\ y = 0$

(H). $\left(D^2+2\,D+5\right)^2\ y = 0$

Problem 8. *(15 points).* Find a particular solution

$$2\,y'' + 3\,y' - 5\,y = 2\,x^3 + 1.$$

Problem 9. *(15 points).* Find the *form* of the particular solution given by the method of undetermined coefficients for the ordinary differential equation

$$y'' + 2\,y' + 5\,y = 6\,x + x\mathrm{e}^{-x} + 2\mathrm{e}^{-x}\cos(2x) + 3\,x\mathrm{e}^{x}\sin(2x).$$

Do not solve for the coefficients.

Problem 10. *(20 points).* Use the method of variation of parameters to find a particular solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + 4\,y = 8\,\sec(2x).$$

3 Final Exam

3.1 LAPLACE transform tables from the exam

Laplace Transform Exchange Formulae

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\begin{aligned}
\mathcal{L}\left\{e^{at} f(t)\right\}(s) &= F(s-a) \\
\mathcal{L}\left\{t^n f(t)\right\}(s) &= (-1)^n F^{(n)}(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) &= \int_s^\infty F(r)\,dr \quad \left(\text{if } \frac{f(t)}{t} \text{ integrable at } 0\right) \\
\mathcal{L}\left\{\int_0^t f(r)\,dr\right\}(s) &= \frac{F(s)}{s} \\
\mathcal{L}\left\{\frac{df}{dt}\right\}(s) &= sF(s) - f(0) \quad \left(\text{if } f \text{ cont. on } [0,\infty)\right) \\
\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\}(s) &= s^2 F(s) - sf(0) - f'(0) \quad \left(\text{if } f,\ f' \text{ cont. on } [0,\infty)\right) \\
\mathcal{L}\left\{u(t-a) f(t-a)\right\}(s) &= e^{-as} F(s) \quad \left(u = \text{ unit step}\right) \\
\mathcal{L}\left\{f(at)\right\}(s) &= \frac{1}{a} F\left(\frac{s}{a}\right).
\end{aligned}$$

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$ then $\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)$.

Sample Laplace Transforms

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}\right\}(s) &= \frac{1}{s-a} \\
\mathcal{L}\left\{t^{n}\right\}(s) &= \frac{n!}{s^{n+1}} \\
\mathcal{L}\left\{\cos\omega t\right\}(s) &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}\left\{\sin\omega t\right\}(s) &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}\left\{e^{at}\cos\omega t\right\}(s) &= \frac{s-a}{(s-a)^2+\omega^2} \\
\mathcal{L}\left\{e^{at}\sin\omega t\right\}(s) &= \frac{\omega}{(s-a)^2+\omega^2} \\
\mathcal{L}\left\{\sqrt{t}\right\}(s) &= \frac{\sqrt{\pi}}{2s^{3/2}} \\
\mathcal{L}\left\{t^{n}e^{at}\right\}(s) &= \frac{n!}{(s-a)^{n+1}}.
\end{aligned}
$$

3.2 The exam problems

Problem 11. *(25 points).* A 100 gallon tank contains 100 gallons of brine of concentration 1.0 oz salt per gallon. Brine of concentration 2.0 oz salt per gallon flows into the tank at 2 gallons per minute. The well-mixed solution is pumped out at the same rate. Find the concentration of the brine solution in the tank at the end of 30 minutes.

Problem 12. *(25 points).* Solve the initial value problem

$$\frac{dp}{dt} = e^{-2t}\, p\,(1-p), \quad p(0) = 1/2.$$

Problem 13. *(20 points).* **(A)** Solve the exact ordinary differential equation

$$\left(2xy - \sec^2(x)\right)\, dx + \left(x^2 + 2y + \cos(y)\right)\, dy = 0.$$

(B) The differential equation

$$(2y - 6x)\, dx + \left(3x - 4x^2 y^{-1}\right)\, dy = 0$$

has an integrating factor of the form $\mu = x^p y^q$ where p and q are integers. Find p and q.

Problem 14. *(20 points).* Solve the (homogeneous) ordinary differential equation

$$x^2 \frac{dy}{dx} = xy + y^2 + x^2.$$

Problem 15. *(20 points).* For each of the following linear homogeneous ordinary differential equations find the general solution (fundamental solution) in real form:

(A). $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5\,y = 0$

(B). $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 0$

(C). $\dfrac{d^2y}{dx^2} - 2\,\dfrac{dy}{dx} + y = 0$

(D). $\dfrac{d^2y}{dx^2} + 5\,\dfrac{dy}{dx} + 6\,y = 0$

Problem 16. *(20 points).* Find the *form* of the particular solution given by the method of undetermined coefficients for the ordinary differential equation

$$y'' + 2\,y' - 3\,y = x^3 + x + x^2 e^x + e^x \cos(x) + x^2 e^{-3x} + e^{-3x} \sin(x).$$

Do not solve for the coefficients.

Problem 17. *(20 points).* Use the method of variation of parameters to find a particular solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + y = \sec^3(x).$$

Problem 18. *(25 points).* If $y(t)$ is the solution to the initial value problem

$$3\,\frac{d^2y}{dt^2} - 5\,\frac{dy}{dt} + 13\,y = \cos(2t), \quad y(0) = 3,\; y'(0) = -2,$$

find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of y. Note it is not necessary to solve the initial value problem.

Problem 19. *(25 points).* Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{6\,s^2 - s - 17}{(s+2)(s^2-1)}\right\}.$$

4 Contact Information

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