Applied Differential Equations – Mth 256

Archive – Spring 1997 Files

Oct 8, 2000

This archive contains the sample problems, two quizzes and the final exam from Mth 256 Spring 1997. The original test instructions, headers and formatting have not been preserved. There is a total of 183 problems below. Be sure to let me know if you require more problems.

Contents

1 Sample Problems – Part 1

The 117 problems below are mostly extracted from old assignments and old tests. There may be some overlap with my old tests archive.

One way to determine the volume of blood in a person is to remove the blood, measure it, and then replace it. Needless to say, this procedure is a bit inconvenient for the person involved. Therefore one wants to have an indirect, less invasive method. Complicated variations on the following simpler problem may be used.

Problem 1. Consider a tank of volume V which is full of, for example, water. Assume water runs into the tank at a rate of α liters/minute and drains out of an opening in the bottom at the same rate. At a certain time t_0 we inject Q_0 grams of dye into the inflow. Assume that the dye instantly enters the tank and is uniformly well–mixed throughout the tank at all successive times. If Q is the amount of dye in the tank at time t then

$$\frac{dQ}{dt} = -\alpha \frac{Q}{V}, \quad Q(t_0) = Q_0.$$

At a certain time $t_1 > t_0$ the concentration of dye in the outflow is found to be β grams/liter. Another measurement 3.5 minutes later is found to yield a concentration of $\beta/2$ grams/liter. If $\alpha = 2$ liters/minute find the volume of the tank.

Consider a tank initially full of some fluid. Assume that the tank is drained through an orifice in the bottom. According to TORICELLI (1608–1647) the fluid issues from the orifice with a velocity u given by $u = \sqrt{2gh}$, where $g = 32.2$ ft/sec^2 is the acceleration of gravity and h (the head) is the instantaneous height of the fluid surface above the orifice. Thus if a is the cross–sectional area of the stream of fluid issuing from the orifice then

$$\frac{dV}{dt} = -au$$

where V is the instantaneous volume of fluid in the tank. If $A(h)$ is the cross–sectional area of the tank at height h then from calculus we know that

$$\frac{dV}{dt} = A(h)\,\frac{dh}{dt}.$$

Combining the two facts above we obtain the following differential equation for h:

$$A(h)\,\frac{dh}{dt} = -a\,\sqrt{2gh}.$$

Remark: According to BORDA, $a = BA$ where A is area of the orifice and B is BORDA's constant. B depends on the fluid – for water we have approximately $B = 0.6$.

Problem 2. Two conical tanks (right circular cones) of the same size are full of water. On tank is upside–down (compared to the other). Both are drained through an orifice near the lower end. Both orifices are the same size. Both tanks are initially full. Whch tank drains first? Find the ratio of the length of time that it takes to drain the tanks.

Problem 3. Two tanks, one conical, the other cylindrical, with the same base radius R and the same height H are initially full of fluid. Both tanks are drained through an orifice in the bottom. Both orifices have the same cross–sectional area. Let T_1 be the length of time it takes to drain the conical tank and let T_2 be the length of time it takes to drain the cylindrical tank. Compute T_1/T_2.

The ratio of the volumes is $1/3$. This fact is of course totally irrelevant.

Here's a problem that's just *food for thought*. If you think there are three solutions then you are on the right track, but you will need to think some more, because i fact, there's only one.

Problem 4. Let $y(t)$ be a nonconstant solution of the ordinary differential equation

$$\frac{dy}{dt} = y^2 + y - 2.$$

Assume $y(t)$ has an inflection point at $t = t_1$. Without solving the ordinary differential equation compute $y(t_1)$.

After all that thought you are probably ready for some dull and boring drill problems.

Problem 5.

$$\frac{dy}{dt} = 4 + y^2$$

Problem 6.

$$\frac{dy}{dx} = \cos y$$

Problem 7.

$$\frac{dy}{dt} = y^2 - 3y + 2$$

Problem 8.

$$\frac{dy}{dx} = y^2 - 2y + 1$$

Problem 9.

$$\frac{dx}{dt} = \frac{t^2}{x}$$

Problem 10.

$$\frac{dy}{dx} = \frac{x^2}{y(1 + x^3)}$$

Here's some equally dull drill initial value problems. Solve each problem *exactly*. Don't use a calculator to evaluate constants. The practice in manipulating exponentials, logarithms, and so on, is invaluable. Don't pass it up.

Problem 11.

$$\frac{dy}{dx} = \frac{xy + y}{x}, \quad y(1) = -2.$$

Problem 12.

$$\frac{dy}{dx} = \frac{xy + y}{x} \log(x), \quad y(1) = -2.$$

Here $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 13.

$$\frac{dy}{dx} + 2xy = 0, \quad y(0) = -2$$

Problem 14. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} \log(x), \quad y(\mathrm{e}) = \mathrm{e}$$

where e is the base of the natural logarithm (EULER's number) and $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 15.

$$x\frac{dy}{dx} + y = 0, \quad y(1) = 4$$

Problem 16.

$$x\frac{dy}{dx} - 4y = 0, \quad y(2) = -8$$

Problem 17.

$$\frac{dy}{dt} + (\tan t)y = 0, \quad y(0) = 2$$

Problem 18.

$$\frac{dy}{dt} + \frac{y}{t} = 0, \quad y(1) = -3$$

Problem 19.

$$x^2\frac{dy}{dx} + y = 0, \quad y(1) = 1$$

Problem 20.

$$\frac{dy}{dt} - \frac{y}{t} = 0, \quad y(1) = -3$$

Problem 21.

$$\frac{dy}{dx} + 4x^3y = 0, \quad y(1) = 2$$

Problem 22.

$$\frac{dy}{dx} + (\log x)y = 0, \quad y(2) = 1$$

Here $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 23.

$$x\frac{dy}{dx} + (x+1)y = 0, \quad y(2) = \mathrm{e}^{-2}$$

Problem 24.

$$(x^2+1)\frac{dy}{dx} - xy = 0, \quad y(\sqrt{3}) = -1$$

Problem 25.

$$(x^2+4)\frac{dy}{dx} + y = 0, \quad y(\pi/2) = \sqrt{e}$$

Problem 26.

$$\frac{dy}{dx} + (\cos x)y = 0, \quad y(0) = 2$$

Problem 27.

$$\frac{dy}{dx} + (3\sin x)y = 0, \quad y(0) = 1/2$$

Problem 28.

$$\frac{dy}{dx} + (6x^2+2x)y = 0, \quad y(1) = 1$$

Problem 29.

$$\frac{dy}{dx} - (\cot x)y = 0, \quad y(\pi/2) = 1$$

Problem 30.

$$\frac{dy}{dx} + y\cos(x) = \cos(x), \quad y(0) = 3.$$

Problem 31. Solve the initial value problem

$$\frac{dp}{dt} = \mathrm{e}^{-2t}\, p(1-p), \quad p(0) = 0.5.$$

Find

$$\lim_{t\to\infty} p(t).$$

Unless I mistyped something all of the ordinary differential equations above are separable. Here's some more dull drill problems where we throw in some linear ordinary differential equations as well. Some of the these problems are both linear and separable. In those cases you should solve the problem in two ways: one by separating the variables, the other by using an integrating factor.

Problem 32.

$$\frac{dy}{dx} = -2y + x$$

Problem 33.

$$\frac{dy}{dx} = -2y + 1$$

Problem 34.

$$\frac{dy}{dx} = -2y$$

Problem 35.

$$\frac{dy}{dx} = \frac{y}{x-3} + x^2$$

Problem 36.

$$\frac{dy}{dx} = xy + x$$

Problem 37.

$$\frac{dy}{dx} = (\cot x)y + \sin x$$

Problem 38.

$$\frac{dy}{dx} = \frac{y}{x} + \sin(x^2)$$

Problem 39.

$$\frac{dy}{dx} = -\frac{y}{x} + 3y$$

Problem 40.

$$\frac{dy}{dx} = -\frac{y}{x} + 3x$$

Problem 41.

$$\frac{dy}{dx} = -\frac{y}{x} + 3$$

Problem 42.

$$\frac{dy}{dx} - 2xy = x, \quad y(0) = 1$$

Problem 43.

$$\frac{dy}{dx} - 2xy = 1, \quad y(0) = 1$$

(Write the solution as a definite integral.)

Problem 44.

$$x\frac{dy}{dx} + 2y = 4x^2, \quad y(1) = 2$$

Problem 45.

$$\frac{dy}{dx} + \frac{y}{x} = \sin x$$

Problem 46.

$$\frac{dy}{dx} + (\tan x)y = x\sin(2x)$$

Problem 47.

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Problem 48.

$$\frac{dy}{dx} = \frac{x^3 - 2y}{x}$$

Problem 49.

$$\frac{dy}{dx} + y = \mathrm{e}^{-x}$$

Problem 50.

$$\frac{dy}{dx} + (\sec x)y = 0$$

Problem 51.

$$\frac{dy}{dx} + 3x^2 y = 0$$

Problem 52.

$$\frac{dy}{dx} - (\log x)y = 0, \quad y(1) = -2$$

Here $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 53.

$$\frac{dy}{dx} + y = x + 1$$

Problem 54.

$$\frac{dy}{dx} + 2xy = 8x^3$$

Problem 55.

$$\frac{dy}{dx} - \frac{y}{x} = x^2 + 2x - 1$$

Assume $x > 0$.

Problem 56.

$$\frac{dy}{dx} + \frac{y}{x} = 8x^3 + 1$$

Assume $x > 0$.

Problem 57.

$$\frac{dy}{dx} + \frac{(x+1)y}{x} = 1$$

Assume $x > 0$.

Problem 58.

$$\frac{dy}{dx} - \frac{y}{x-1} = x^3 + 3x - 2$$

Assume $x > 1$.

Problem 59.

$$xy\frac{dy}{dx} = x^2 + 1$$

Problem 60.

$$y\frac{dy}{dx} = x\,e^{x^2+y^2}.$$

Problem 61.

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{1}{x}$$

Assume $x > 0$.

That ought to be enough drill for now. How about some *food for thought*?

Problem 62. Let $a > 0$ and $b > 0$ be constants. Show that the (curious) initial value problem

$$\frac{dy}{dt} + ay = \int_0^b y(s)\,ds, \quad y(0) = 1$$

has a *unique* solution if and only if $a^2 - ab - e^{-ab} + 1 \neq 0$ and infinitely many solutions otherwise.

Problem 63. Find a solution of

$$\frac{dy}{dt} + y\tan t = 0, \quad \int_0^{\pi/6} y(s)ds = 2.$$

Problem 64. Let u be a continuously differentiable function on $[0, \infty)$. Let M and k be constants, $k \neq 0$. Suppose we have

$$\frac{du}{dt} - ku \leq 0 \quad \text{for all } t \geq 0.$$

If $u(0) = 0$ then show that

$$u(t) \leq \frac{M}{k}\left(e^{kt} - 1\right) \quad \text{for all } t \geq 0.$$

Here's a hint: Think about the integrating factor.

Problem 65. Find a function $y(t)$ such that

$$y(t) + \int_0^t y(s)\,ds = t^2$$

for all $t \in \mathbb{R}$.

Problem 66. A radioactive substance decays to 85% of its original mass in 36 hours. Find the half–life. Find the third–life.

Here are some problems involving integrating factors and exact ordinary differential equations.

Problem 67. For what value of k is $\left(x^2+y^2\right)^k$ an integrating factor for

$$-y\,dx + x\,dy = 0?$$

Problem 68. For what values of p and q is $x^p y^q$ an integrating factor for the ordinary differential equation

$$(6y^2 + 3y - 4xy)\,dx + (-3x^2 + 3x + 8xy)\,dy = 0?$$

Problem 69. The differential equation

$$\left(y - xy^2\right)\,dx + \left(x + x^2y^2\right)\,dy = 0$$

has an integrating factor of the form $x^m y^n$. Find the integrating factor. Then solve the differential equation.

Problem 70. Find the general solution of the ordinary differential equation

$$(1 + \log(xy))\,dx + \left(1 + \frac{x}{y}\right)\,dy = 0.$$

Problem 71. Solve the initial value problem

$$\left(3x^2y^2 - 2xy^3 - 2x - 1\right)\,dx + \left(2x^3y - 3x^2y^2 - 8y^3 - y + 1\right)\,dy = 0, \quad y(0) = 2.$$

Problem 72. Find an integrating factor which depends only on y and then solve the differential equation

$$(2y + y^2 - 6xy)\,dx \;+\; (4x + 3xy - 6x^2)\,dy = 0.$$

Problem 73. The ordinary differential equation

$$\left(6\,xy + y^2\right)\,dx + \left(9\,x^2 - 6 + 4\,xy\right)\,dy = 0$$

has an integrating factor μ depending only on y. Find the integrating factor and then solve the ordinary differential equation.

Problem 74. For what values of m and n is $x^m y^n$ an integrating factor for the differential equation

$$(3y + 12xy^2)\,dx + (4x + 15x^2y)\,dy = 0.$$

Problem 75. Solve the initial value problem

$$(4x + 4y + 3)\,dx + (4x - 6y - 2)\,dy = 0, \qquad y(2) = 1.$$

Problem 76. Solve the *exact* ordinary differential equation

$$(2xy^3 - y^2 - 2)\,dx + (3x^2y^2 - 2xy + 3)\,dy = 0.$$

Problem 77. The ordinary differential equation

$$(6\,y - 4\,y^2)\,dx + (9\,x - 8\,x\,y)\,dy = 0$$

has an integrating factor of the form

$$\mu = x^m\,y^n.$$

Find m and n and then solve the given ordinary differential equation.

Problem 78. The ordinary differential equation

$$(9\,y^2 + 18\,x^2y^2 + 4\,xy^3 + x)\,dx + (18\,xy + 6\,y^2)\,dy = 0$$

has an integrating factor depending only on x. Find this integrating factor and then solve the given ordinary differential equation.

Here are some problems involving substitution.

Problem 79. Substitute $v = g(y)$ in the ordinary differential equation

$$g'(y)\frac{dy}{dt} + a(t)g(y) = b(t)$$

to obtain a linear ordinary differential equation for v. Now use this technique to solve

$$\mathrm{e}^y\frac{dy}{dt} + \mathrm{e}^y = \mathrm{e}^t.$$

Problem 80. If $n = 0$ or $n = 1$ then the BERNOULLI equation

$$\frac{dy}{dt} + a(t)y = b(t)y^n$$

is linear, and we can solve it. In all other cases, if we multiply it by $(1-n)y^{-n}$ we can use the trick in the previous problem and set $v = y^{1-n}$ to obtain a linear ordinary differential equation for v. Use this idea to solve the BERNOULLI equation

$$\frac{dy}{dt} - y = ty^2.$$

Problem 81. Let a and $b \neq 0$ be constants. If

$$\frac{dy}{dx} = f(ax + by)$$

show that the substitution $z = ax + by$ produces a separable ordinary differential equation for z. Use this method to solve the initial value problem

$$\frac{dy}{dx} = (x+y)^2, \quad y(0) = 0.$$

Problem 82. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \big(x + y + 2\big)\big(x + y\big).$$

Problem 83. Solve the initial value problem

$$\frac{dy}{dx} = 2 - (2x - y)^2, \quad y(0) = 1.$$

Problem 84. Use the substitution $w = 2\,x + 3\,y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(2x+3y)^2 - 4\,(2x+3y) + 4}{6\,(2x+3y)}.$$

Problem 85. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(x+y-1)\,(x+y)}{2x+2y+1}$$

Problem 86. Recall the substitution $y = tv$ can be used to solve the homogeneous ordinary differential equation

$$\frac{dy}{dt} = h(y/t).$$

Use this method to solve the ordinary differential equation

$$\frac{dy}{dt} = \frac{y - 4t}{t - y}.$$

Problem 87.

$$\frac{dy}{dt} = \frac{y^2 + 2ty}{t^2}$$

Problem 88.

$$\frac{dy}{dt} = \frac{t^2 + y^2}{t^2}$$

It is time for a bit more *food for thought*.

Problem 89. Show that $y(t) = 0$ and $y(t) = t^3$ are solutions of the initial value problem

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(0) = 0.$$

Why does this example not contradict the uniqueness theorem?

Problem 90. Let's assume you have solve the ordinary differential equation

$$\frac{dy}{dx} = 2xy^2$$

and that you obtained the one–parameter family of solutions

$$y(t) = \frac{1}{c - x^2}.$$

Now if we want to select c so we have a solution satisfying the initial condition $y(0) = 0$ we have a problem. What is the solution to this initial value problem? Can we obtain the solution to the initial value problem by letting the parameter c pass to a suitable limit?

Mixing problems are a nice application of linear first order ordinary differential equations. Many very elaborate variations on the basic theme occur in biological applications. Here's a few fairly straight-forward mixing problems.

Problem 91. A 50 liter tank initially contains 10 liters of brine of concentration 0.5 gram/liter salt. A brine solution containing 1 gram/liter salt runs into the tank at the rate 4 liter/min. and the well-stired solution is drained off at the rate 2 liter/min. Find the concentration of salt in the brine in the tank at the onset of overflow.

Problem 92. A large tank contains 80 gallons of brine of concentration 1.621 oz/gal salt. Brine of concentration 2.121 oz/gal salt flows into the tank at 3 gal/min. The well–mixed solution is drawn off at the rate of 4 gal/min. When will the brine in the tank reach a *concentration* of 2.013 oz/gal salt?

Problem 93. A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well–mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

Problem 94. A large tank initially contains 100 L of brine of concentration 0.6 g/L salt. Brine of concentration 2.1 g/L runs into the tank at 6.0 L/min and the well–mixed solution is drained off at 4.0 L/min. Find the concentration of salt in the tank at the moment that the tank contains 220 L brine.

Problem 95. A 200 gal tank initially contains 100 gal fresh water. Brine of concentration 1.2 oz/gal salt flows into the tank at the rate 3 gal/min. The well–mixed solution is drawn off at the rate 2 gal/min. Find the *concentration* of salt in the tank at the very moment that it begins to overflow.

Problem 96. A 200 L tank initially contains 100 L of brine of concentration 3 g/L salt (i.e., 3 grams salt per liter water). Brine of concentration 5 g/L salt runs into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the solution in the tank at the moment that the tank begins to overflow.

Problem 97. A brine solution consisting of 0.06 oz/gal salt dissolved in water flows into a large tank at the rate 3.0 gal/min. The solution inside the tank is kept well–mixed and flows out of the tank at the rate 2.0 gal/min. If the tank initially contains 50.0 gal of brine of concentration 0.03 oz/gal determine the amount of salt in the tank after t minutes. When will the concentration of salt in the tank reach 0.05 oz/gal? Assume the tank is so large that it does not overflow.

Problem 98. A tank of volume 400 L initially contains 200 L of brine of concentration $\frac{1}{5}$ g/L of salt. Brine of concentration 1 g/L flows into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the tank at the moment that it overflows.

Problem 99. A tank initially contains 200 L of brine of concentration $\frac{1}{5}$ g/L of salt. Brine of concentration $\frac{1}{2}$ g/L flows into the tank at 2 L/min. The well–mixed solution is drawn off at the rate 3 L/min. Find the *concentration* of salt in the tank after 100 minutes.

NEWTON's law of cooling (or heating) states that when a body at temperature T is immersed in a medium at temperature A then the rate of change of T is proportional to the difference in the temperatures. Explicitly

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant.

NEWTON's law of cooling is a useful application of first order ordinary differential equations.

Problem 100. A cup of coffee initially at temperature $T_0 = 190°$ F is brought into a room at temperature $A = 65°$ F. The heat capacity of the room (compared to the coffee) is so large that we may regard A as being constant. After 2 minutes the temperature of the coffee is $145°$ F. What temperature will the coffee be an additional minute later.

Problem 101. A thermometer reading $92°$ F is immersed in a cooler liquid. After 3 seconds the thermometer reads $80°$ F. Another 3 seconds later it reads $76°$ F. What is the temperature of the fluid?

Problem 102. Consider a cup of coffee in a room of temperature $A = 70°$ F. The cup is sitting on a small heating pad which is supposed to keep the coffee warm. If T is the temperature of the coffee then

$$\frac{dT}{dt} = -k\,(T - A) + U$$

where U is a constant depending on the heater and the cup and k is a constant depending on the cup. Initially the temperature of the coffee is $183°$ F, 5 minutes later the temperature is $155°$ F, and an additional 5 minutes later the temperature is $135°$ F.

Find the temperature T as a function of time. Compute

$$\lim_{t\to\infty} T(t).$$

Problem 103. A thermometer initially reading $62°$ F is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads $167°$ F. After an additional 1 second it reads $179°$ F. If A denotes the temperature of the coffee, T denotes the temperature reading of the thermometer and t denotes time in seconds then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. We regard the temperature A of the coffee also as constant. Find the temperature of the coffee.

Problem 104. A thermometer is brought into a certain room. The room has temperature $A = 25°$ C. If T is the temperature displayed by the thermometer then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads $21.4°$ C. An additional 20 seconds later it reads $23.4°$ C. What was the initial reading on the thermometer at the time that it was first brought into the room?

Problem 105. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing sinusoidally, say

$$A = A_0 + A_1 \cos(\omega t)$$

where A_0, A_1 and $\omega > 0$ are constants, and t is time. According to NEWTON's law of cooling we have

$$\frac{dT}{dt} + kT = kA_0 + kA_1 \cos(\omega t)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1, ω and k. (Do not neglect the arbitrary constant.) Which part of your solution represents the steady–state? What is the amplitude, period and phase of the steady–state solution?

Problem 106. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing linearly (for a while at least), say

$$A = A_0 + A_1 t$$

where A_0 and A_1 are constants, and t is time. According to NEWTON's law of cooling we have

$$\frac{dT}{dt} + kT = kA_0 + kA_1 t$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1 and k. (Do not neglect the arbitrary constant.)

Problem 107. Consider a cup of coffee in a room of temperature A. Initially the temperature of the coffee is 183° F, 3 minutes later the temperature is 155° F, and an additional 3 minutes later the temperature is 135° F. Find the temperature T as a function of time. Compute the temperature of the room.

Problems of finding orthogonal trajectories show a bit of the usefullness of ordinary differential equations in geometry. If we have a one–parameter family of curves

$$F(x, y, \alpha) = 0,$$

differentiate y implicitly with respect to x

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

and then eliminate α between these two equations, we obtain an expression of the form

$$\frac{dy}{dx} = f(x, y).$$

Apart from some pathologies the general solution to this first order ordinary differential equation is the one–parameter family of curves we started with. It follows that the orthogonal trajectories, the curves orthogonal to each curve in our original family, are the solutions to the ordinary differential equation

$$\frac{dy}{dx} = \frac{-1}{f(x, y)}.$$

Problem 108. Find the family of orthogonal trajectories to the one–parameter family of hyperbolas given by $2y^2 - x^2 = \alpha$.

Problem 109. Find the orthogonal trajectories for the family of ellipses

$$2y^2 + x^2 = \alpha \quad (\alpha \text{ is an arbitrary parameter}).$$

Problem 110. Find the family of orthogonal trajectories to the one–parameter family of cubics

$$y = \alpha x^3, \quad \alpha = \text{ arbitrary parameter}.$$

Problem 111. Consider the 1–parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1$$

Find the 1–parameter family of orthogonal trajectories.

Problem 112. Given the one–parameter family of curves

$$x^2 + y^2 = \alpha + 2\,\log(x) \quad (\alpha \text{ is the parameter})$$

find the one–parameter family of orthogonal trajectories.

Problem 113. Given the 1–parameter family of curves

$$y^2 = \alpha\, \mathrm{e}^{x^2+y^2}, \; (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

Problem 114. Given the 1–parameter family of curves

$$x^2 + \alpha\, y^2 = \alpha, \; (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

Pursuit problems are interesting application of ordinary differential equations. Unfortunately most pursuit problems can not be solved in closed form in terms of elementary functions.

We will consider a not-too-bright (or perhaps, just well-fed) dog chasing a rabbit. If you prefer you can consider a missile chasing an aircraft. The not-too-bright part refers to the dog's proclivity to head directly towards the rabbit rather than heading it off. In practice, real dogs are actually much smarter.

Consider then a rabbit hopping along a trajectory $(p(t), q(t))$. At a certain time t_0 the dog sees the rabbit and gives chase. The dog's stragety is to run directly towards the rabbit a full (and constant) speed u. The rabbit, not too concerned, is hopping along at a constant speed v. Assume in our coordinate system, the dog's motion is strictly towards the left, so if (x, y) is the position of the dog, then

$$\frac{dx}{dt} < 0.$$

In this case we can parametrize the dog's trajectory by x. Since the dog runs directly towards the rabbit at all times we have

$$\frac{dy}{dx} = \frac{y - q}{x - p}.$$

If we introduce the slope $w = \dfrac{dy}{dx}$ of the dog's trajectory we have

$$(x - p)w = y - q.$$

Now if we differentiate with respect to x and simplify we obtain

$$(x - p)\,\frac{dw}{dx} = -\left(\frac{dq}{dt} - w\frac{dp}{dt}\right)\,\frac{dt}{dx}.$$

From calculus we know

$$u = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} \left|\frac{dx}{dt}\right| = -\left(1 + w^2\right)^{1/2}\,\frac{dx}{dt}.$$

It follows that

$$\frac{dt}{dx} = -\frac{1}{u}\left(1 + w^2\right)^{1/2}$$

Substituting this expression above we obtain the equation

$$(x - p)\frac{dw}{dx} = \left(\frac{dq}{dt} - w\frac{dp}{dt}\right)\,\left(1 + w^2\right)^{1/2}.$$

Problem 115. Suppose the rabbit (in the discussion above) runs along the y–axis, so $p(t) = 0$ and $q(t) = b + tv$. Assume $t_0 = 0$ and $b > 0$. Then the equation for w becomes

$$x\,\frac{dw}{dx} = \frac{v}{u}\,\left(1 + w^2\right)^{1/2},$$

a separable first order ordinary differential equation. Assume the dog starts at the point $(a, 0)$ where $a > 0$. Then we have the initial condition

$$w = -\frac{b}{a} \quad \text{when} \quad x = a.$$

Once we have found w then we solve the initial value problem

$$\frac{dy}{dx} = w, \quad y = b \quad \text{when} \quad x = a.$$

Show if $u > v$ then the dog catches the rabbit at time

$$t_1 = \frac{vb + u(a^2 + b^2)^{1/2}}{u^2 - v^2}.$$

Problem 116. A river of width a is flowing at a constant speed v. At a certain time a boat sets sail from one bank and maintains a heading towards a lighthouse initially directly opposite on the other bank. The boat sails at a constant speed u. Under what conditions will the boat land on the opposite bank at the lighthouse and how long will it take?

Hint: One way to approach this problem is to use moving coordinates, i.e., think of the river as stationary and the lighthouse as moving upstream. Now think about dogs and rabbits.

Remark: If the goal is simply to reach the other bank then that can be achieved by keeping a heading perpendicular to the bank. But if one insists on landing at the lighthouse, then it is not always possible. Is there a lesson hidden in this?

As our final example let's consider the parachute equation. Consider an object falling through the atmosphere without tumbling. Let the height be y so $v = -\dfrac{dy}{dt}$ is the downward component of the velocity. Assume that the acceleration of gravity g is a constant (about 32.2 ft/sec^2) and that the mass of the object is m. If we assume that atmospheric drag is given by kv^2 where $k > 0$ is a constant (depending on the size, shape and aspect of the body, on air resistance, etc.) and we assume that bouyancy is negligible, then NEWTON's law of motion gives

$$\frac{dv}{dt} = g - \frac{k}{m} v^2.$$

The equation of motion has an equilibrium (i.e., constant) solution, namely

$$v(t) = \alpha = \left(\frac{mg}{k}\right)^{1/2}.$$

If we introduce the "dimensionless" speed $u = v/\alpha$ we obtain

$$\frac{du}{dt} = \beta\,(1 - u^2) \quad \text{where } \beta = \frac{g}{\alpha}.$$

Problem 117. Solve the ordinary differential equation above for u and do a little algebra to show

$$u(t) = \frac{1}{\beta}\,\frac{h'(t)}{h(t)}$$

where

$$h(t) = u_0\left(e^{\beta t} - e^{-\beta t}\right) + e^{\beta t} + e^{-\beta t}$$

where $u_0 = u(0) = v_0/\alpha$. Conclude that

$$y_0 - y(t) = \frac{\alpha^2}{g} \log\left(\frac{v_0}{\alpha}\sinh\left(\frac{gt}{\alpha}\right) + \cosh\left(\frac{gt}{\alpha}\right)\right).$$

Then show $\lim_{t\to\infty} v(t) = \alpha$. Show that for large t

$$y_0 - y(t) \approx \alpha t - (\log 2)\frac{\alpha^2}{g}$$

if $v_0 = 0$ (i.e., the object falls from rest).

If a man and his equipment, weighing a total of 260 lbs. fall, from rest in free fall, and falls 25,100 feet in the first 100 seconds and, also from rest 29,300 feet in 116 seconds, determine α (ft/sec) and k (slugs/ft). Note these are real numbers determined by the U.S. army in 1942.

2 Sample Problems – Part 2

Problem 118. *Food for thought.* Recall the pendulum equation for a pendulum of length ℓ

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$

If we multiply this equation by $\dfrac{d\theta}{dt}$ and integrate we obtain the first order ordinary differential equation

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{\ell}\cos\theta = K,$$

where K is related to the total energy E by $E = Km\ell^2$. It is not obvious, but the solutions of this equation are periodic. Assuming initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = 0$ and a period T we have

$$\frac{d\theta}{dt} = -\left(\frac{2g}{\ell}\right)^{1/2}(\cos\theta - \cos\theta_0)^{1/2}, \quad \text{for } 0 \le t \le T/4.$$

Show that

$$\begin{aligned} T &= 4\left(\frac{\ell}{2g}\right)^{1/2}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\ &= 2\left(\frac{\ell}{g}\right)^{1/2}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}. \end{aligned}$$

Now let $k = \sin(\theta_0/2)$ and make the change of variable $\sin(\theta/2) = k\sin(\alpha)$. Show that

$$T = 4\left(\frac{\ell}{g}\right)^{1/2}\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}} = 2\pi\left(\frac{\ell}{g}\right)^{1/2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots\right].$$

Thus the period T actually does depend on the initial displacement θ_0.

Problem 119. *Here is a problem that I should have put on the first problem set.* A charred branch of a tree killed by the eruption of Mount Mazama that formed Crater Lake was estimated to contain about 45% of the original amount of carbon-14. Given that the half–life of carbon-14 is about 5568 years, date the eruption.

Problem 120. *More food for thought.* Consider a linear spring, that is, a spring satisfying Hooke's hypothesis. Suppose the spring has spring constant k and one end is attached to a frictionless trolley of mass m and the other end is attached to a piston moving horizontally sinusoidally. If the displacement of the piston at time t is $a\sin(\omega t)$, and x is the displacement of the trolley from its equilibrium position (with respect to the midpoint position for the piston), then the spring is stretched by the amount $x - a\sin(\omega t)$. Thus the equation of motion is

$$m\frac{d^2x}{dt^2} + k(x - a\sin(\omega t)) = 0.$$

If we let $\omega_0 = \sqrt{k/m}$ be the natural frequency for this spring–mass combination then we have

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \omega_0^2 a\sin(\omega t).$$

If $\omega \neq \omega_0$ find the general solution. If we have the (carefully contrived) initial conditions $x(0) = 0$ and $x'(0) = -\omega_0^2 a/(\omega + \omega_0)$ show that

$$x(t) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2}\left[\sin(\omega t) - \sin(\omega_0 t)\right].$$

Since

$$\omega = \frac{\omega + \omega_0}{2} + \frac{\omega - \omega_0}{2}, \quad \text{and } \omega_0 = \frac{\omega + \omega_0}{2} - \frac{\omega - \omega_0}{2}$$

a little trigonometry shows that

$$x(t) = A(t)\sin\left(\frac{\omega + \omega_0}{2}t + \frac{\pi}{2}\right)$$

where

$$A(t) = \frac{2\omega_0^2 a}{\omega_0^2 - \omega^2}\sin\left(\frac{\omega - \omega_0}{2}t\right).$$

If ω is close to ω_0 then $\dfrac{\omega + \omega_0}{2} \sim \omega_0$ and so we may regard the motion as being roughly of frequency ω_0, but with amplitude $A(t)$ varying slowly with frequency $\dfrac{\omega - \omega_0}{2}$. This is the phenomenon of beats. Other initial conditions give the same result but with slightly more complicated $A(t)$ and with the phase ($\pi/2$ above) not constant, but also varying slowly. Thus the actual motion is very complicated, but our view of it, thanks to trigonometry, is relatively simple.

Problem 121. The ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(4x^2 + 2\right)y = 4x^3\cos(2x)$$

has particular solutions $y = \phi_j(x), \;\; j = 1, 2, 3$ given by

$$\begin{aligned} \phi_1(x) &= x^2 \sin(2x) + 2x \sin(2x) + x \cos(2x) \\ \phi_2(x) &= x^2 \sin(2x) - 2x \sin(2x) + x \cos(2x) \\ \phi_3(x) &= x^2 \sin(2x) + 2x \sin(2x) + 2x \cos(2x). \end{aligned}$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = -\pi^2.$$

The problems below are mostly extracted from old assignments and old tests. There may be some overlap with my old tests archive.

Problem 122. For what value of λ is $y = x^\lambda \log(x)$ a solution of the ordinary differential equation $x^2 y'' - 5xy' + 9y = 0$?

Problem 123. The ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{1/2}, \quad x > 0$$

has a particular solution of the form $Ax^{1/2}$ where A is a constant. **(A)** Find the constant A. **(B)** Does the ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{-1}, \quad x > 0$$

have a solution of the form Ax^{-1}? Explain your answer.

Problem 124. The ordinary differential equation

$$\left(x^3 - x^2\right) \frac{d^2y}{dx^2} - \left(x^3 + 2\,x^2 - 2\,x\right) \frac{dy}{dx} + 2\,\left(x^2 + x - 1\right)\, y = 0$$

has the solution $y_1(x) = x\,\mathrm{e}^x$. Use reduction of order to find a solution y_2 such that $\{y_1,\, y_2\}$ is a fundamental solution set for the given ordinary differential equation.

Problem 125. The ordinary differential equation

$$x\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} - y = 0$$

has the solution $y_1(x) = x\,\mathrm{e}^x$. Use reduction of order to find a solution y_2 such that $\{y_1,\, y_2\}$ is a fundamental solution set for the given ordinary differential equation.

Problem 126. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $x^2 \dfrac{d^2y}{dx^2} + 11\,x\,\dfrac{dy}{dx} + 169\,y = 0$

Part (B): $\dfrac{d^2y}{dx^2} + 4\,\dfrac{dy}{dx} + 5\,y = 0$

Part (C): $x^2 \dfrac{d^2y}{dx^2} - 5\,x\,\dfrac{dy}{dx} + 9\,y = 0.$

Problem 127. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2\,y = 0$

Part (B): $4\,\dfrac{d^2y}{dx^2} + 4\,\dfrac{dy}{dx} - 3\,y = 0$

Part (C): $4\,\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + y = 0$

Problem 128. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $x^2\dfrac{d^2y}{dx^2} + 5x\,\dfrac{dy}{dx} + 13\,y = 0$

Part (B): $4\,x^2\dfrac{d^2y}{dx^2} + y = 0$

Part (C): $4\,x^2\dfrac{d^2y}{dx^2} + 8x\,\dfrac{dy}{dx} - 3\,y = 0$

Problem 129. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $\dfrac{d^2y}{dx^2} + 4\,\dfrac{dy}{dx} + 13\,y = 0$

Part (B): $4\,\dfrac{d^2y}{dx^2} + 9\,y = 0$

Part (C): $x^2\dfrac{d^2y}{dx^2} - x\,\dfrac{dy}{dx} + y = 0$

Problem 130. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Part (B): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0$

Part (C): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

Part (D): $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = 0$

Part (E): $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 3y = 0$

Part (F): $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 5y = 0$

Part (G): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 0$

Part (H): $\dfrac{d^2y}{dx^2} + 4y = 0$

Part (I): $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0$

Problem 131. Find the general solution **(in real form)**:

Part (A): $\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} = 0$

Part (B): $\dfrac{d^2y}{dx^2} - 4\,y = 0$

Part (C): $\dfrac{d^2y}{dx^2} + 4\,y = 0$

Part (D): $\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + 4\,y = 0$

Part (E): $\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + 13\,y = 0$

Part (F): $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6\,y = 0$

Problem 132. Find the general solution **(in real form)**:

Part (A): $x^2\,\dfrac{d^2y}{dx^2} - 8x\,\dfrac{dy}{dx} = 0$

Part (B): $x^2\,\dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} - 9\,y = 0$

Part (C): $x^2\,\dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} + 9\,y = 0$

Part (D): $x^2\,\dfrac{d^2y}{dx^2} - 5x\,\dfrac{dy}{dx} + 9\,y = 0$

Part (E): $x^2\,\dfrac{d^2y}{dx^2} - 5x\,\dfrac{dy}{dx} + 25\,y = 0$

Part (F): $x^2\,\dfrac{d^2y}{dx^2} - 6\,y = 0$

Problem 133. Use variation of parameters to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - y = \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^x}.$$

Problem 134. Use the method of undetermined coefficients to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} + 25\,y = 3\,\mathrm{e}^{3x}\,\cos(4x).$$

Problem 135. The ordinary differential equation

$$(x-1)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + y = \frac{(x-1)^2}{x}$$

has complementary function

$$c_1\,x + c_2\,\mathrm{e}^x.$$

Find a particular solution. What is the general solution?

Problem 136. Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6\,y = x^3\,\mathrm{e}^{2x}.$$

Problem 137. Find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x\left(\mathrm{e}^{2x} + \mathrm{e}^{-2x}\right).$$

Problem 138. Solve the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4\mathrm{e}^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Problem 139. Consider the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(x^2 + 2\right)y = x^3 \sec x, \quad x \geq 0.$$

Given that the complementary function is

$$c_1 x \cos x + c_2 x \sin x$$

use variation of parameters to find the general solution.

Problem 140. Use variation of parameters to find the general solution of the ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} - y = 2\,x + 4\,x^{-1}.$$

Problem 141. Use the method of undetermined coefficients (judicious guessing) to find a particular solution:

Part (A): $\dfrac{d^2y}{dx^2} + 3\,\dfrac{dy}{dx} - 10\,y = x^2\,\mathrm{e}^{2x}$

Part (B): $\dfrac{d^2y}{dx^2} - 6\,\dfrac{dy}{dx} + 25\,y = 4x\,\mathrm{e}^{3x}\,\sin(4x)$

Problem 142. Assume that the acceleration of gravity is 9.8 m/sec^2 so that a 10 kg mass will weigh 98 newtons.

A 10 kg mass is suspended from a spring, stretching it by 0.7 m. The mass is started in motion by pulling it down 0.5 m and releasing it. Assume air resistance has magnitude $90\,\dfrac{dx}{dt}$ newtons where x is the downward displacement of the mass from equilibrium.

Part (A): Find the equation of motion of the mass and solve it using the appropriate initial values.

Part (B): How many times does the mass pass through the equilibrium position after being released?

Problem 143. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $6\frac{d^3y}{dx^3} - 17\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 2y = 0$

Part (B): $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Part (C): $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 10\frac{dy}{dx} - 24y = 0$

Part (D): $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 5y = 0$

Part (E): $2\frac{d^3y}{dx^3} - 1\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Part (F): $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 8\frac{dy}{dx} - 12y = 0$

Part (G): $\frac{d^3y}{dx^3} - 12\frac{d^2y}{dx^2} + 44\frac{dy}{dx} - 48y = 0$

Part (H): $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Part (I): $6\frac{d^4y}{dx^4} - 23\frac{d^3y}{dx^3} + 28\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 2y = 0$

Part (J): $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$

Part (K): $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2y = 0$

Part (L): $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Part (M): $\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 9\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$

Part (N): $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

Problem 144. Find a particular solution for each of the following ordinary differential equations.

Part (A): $6\frac{d^3y}{dx^3} - 17\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 2y = x^2$

Part (B): $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^3 - 2x + 5$

Part (C): $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 10\frac{dy}{dx} - 24y = \mathrm{e}^{-x}$

Part (D): $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 5y = x\mathrm{e}^{x}$

Part (E): $2\frac{d^3y}{dx^3} - 1\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos(x)$

Part (F): $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 8\frac{dy}{dx} - 12y = 4\cos(x) - 3\sin(x)$

Part (G): $\frac{d^3y}{dx^3} - 12\frac{d^2y}{dx^2} + 44\frac{dy}{dx} - 48y = x\cos(x)$

Part (H): $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \mathrm{e}^{2x}\sin(x)$

Part (I): $6\frac{d^4y}{dx^4} - 23\frac{d^3y}{dx^3} + 28\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 2y = 2x + 1$

Part (J): $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = x^2 + \mathrm{e}^{x}$

Part (K): $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2y = x^4$

Part (L): $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \mathrm{e}^{2x}$

Part (M): $\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 9\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 12\cos(x) - 5\sin(x)$

Part (N): $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2\sin(x)$

Problem 145. Find the general solution (*in real form*) for each of the following ordinary differential equations.

Part (A): $x^3 \dfrac{d^3y}{dx^3} + x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

Part (B): $x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} - 8y = 0$

Part (C): $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} = 0$

Part (D): $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0$

Part (E): $x^3 \dfrac{d^3y}{dx^3} - x^2 \dfrac{d^2y}{dx^2} + 10x\dfrac{dy}{dx} = 0$

Part (F): $x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 16x\dfrac{dy}{dx} - 26y = 0$

Part (G): $x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = 0$

Part (H): $x^3 \dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = 0$

Problem 146. Find a particular solution for the ordinary differential equation.

$$x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - y = 2x + 3x^2 - 2x^4$$

3 Sample Problems – Part 3

Problem 147. Use the table to compute the LAPLACE transforms

(A) $\mathcal{L}\left\{t^2 e^{3t}\right\}$ **(B)** $\mathcal{L}\left\{e^{2(t-1)}\right\}$ **(C)** $\mathcal{L}\left\{(t+1)^3\right\}$.

Problem 148. Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{2s^2 + 7s + 3}{(s-1)(s+1)(s+2)}\right\}.$$

Problem 149. Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{s^2-4}{(s+1)(s^2+4)}\right\}.$$

Problem 150. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2\,s^2+3}{s^3+2\,s^2-3\,s}\right\}.$$

Problem 151. Find the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s^2-1)(s+1)}\right\}.$$

Problem 152. If

$$\mathcal{L}\left\{f(t)\right\} = \frac{s^3}{s^4-s+2}$$

compute the LAPLACE transform

$$\mathcal{L}\left\{\mathrm{e}^{-2t} f(t)\right\}.$$

Problem 153. Compute and simplify $\mathcal{L}\left\{\mathrm{e}^{t}\cos t\sin t\right\}$.

Problem 154. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\}.$$

Problem 155. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{(s+1)^2(s-1)}\right\}.$$

Problem 156. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s\,(s^2+4)}\right\}.$$

Problem 157. Compute the inverse Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{5\,s^2+5\,s-6}{s(s+1)(s-2)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{6\,s^2-7\,s+3}{s^2(s-1)}\right\}$$

Problem 158. Consider the initial value problem

$$\frac{d^2y}{dt^2}+3\,\frac{dy}{dt}-4\,y=t\,\mathrm{e}^t, \quad y(0)=-2,\ \ y'(0)=3.$$

Find the Laplace transform of the solution to this initial value problem.

Problem 159. Consider the initial value problem

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+5y=4\mathrm{e}^{-t}, \quad y(0)=2, \quad y'(0)=12.$$

Find the LAPLACE transform of the solution to this initial value problem.

Food for thought $\cdots$

Problem 160. If

$$y(t)+\int_0^t y(r)\,dr=1$$

use the LAPLACE transform to find $y(t)$.

Piecewise continuous functions $\cdots$

Problem 161. If

$$f(t)=\begin{cases}3t, & 0\le t\le 2\\ 6, & 2\le t.\end{cases}$$

compute the LAPLACE transform

$$\mathcal{L}\left\{f(t)\right\}=\int_0^\infty \mathrm{e}^{-st}f(t)\,dt.$$

Problem 162. Find the LAPLACE transform of the solution to the initial value problem

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+3y \;=\; \begin{cases}2 & \text{if } 0<t<1\\ t & \text{if } 1<t<4\\ -1 & \text{if } 4<t\end{cases}$$

$$y(0)=1, \quad y'(0)=-2.$$

(Do not solve the differential equation).

Problem 163. Find the Laplace transform of the solution of the following initial value problem:

$$5\,\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2\,y = \begin{cases} 0 \text{ if } 0 \le t < 2 \\ 2 \text{ if } 2 \le t < 3 \\ 0 \text{ if } 3 \le t \end{cases} \qquad y(0) = -2,\;\; y'(0) = 1.$$

Problem 164. Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{ \mathrm{e}^{-\pi s/2}\,\frac{s-2}{s^2-4s+13} \right\}.$$

4 Test 1

Problem 165. Solve the initial value problems

Part (A): $(1+x)\,\dfrac{dy}{dx} = \sqrt{y}, \quad y(0) = 4$

Part (B): $(1+y)\,\dfrac{dy}{dx} = \sqrt{y}, \quad y(0) = 4$

Problem 166. Find the general solution

Part (A): $\dfrac{dy}{dx} + xy = x - y + 1$

Part (B): $\dfrac{dy}{dx} + y = \mathrm{e}^{2x}y^3$ **Hint:** Try the substitution $w = y^{-2}$.

Problem 167. Find the general solution of the ordinary differential equation

$\big(3x + y^2 - \sin(x+y)\big)\;dx + \big(2xy + y^2 - \sin(x+y) + \mathrm{e}^y\big)\;dy = 0.$

Problem 168. A brine solution of 0.4 kg/L salt flows into a large tank at the rate of 3 L/min. The well-mixed solution is drawn off at the rate of 2 L/min. The tank initially held 100 L of brine of concentration 0.1 kg/L salt. Assuming the tank is so large that it does not overflow compute the length of time that it takes for the brine in the tank to reach a concentration of 0.2 kg/L.

Problem 169. A cup of hot coffee is brought into a room. Assume the temperature of the room remains constant, say $A°$ F. Two minutes later the temperature of the coffee is found to be $115°$ F. Another two minutes later it is found to be 97° F. Yet another two minutes later the temperature of the coffee is found to be $85°$ F. Assuming Newton's law of cooling, find the temperature of the room and also the initial temperature of the coffee.

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5 Test 2

Problem 170. Let L be a second order linear ordinary differential operator with constant coefficients. Suppose we have found functions u, v, w such that $L[u] = L[v] = 0$, $L[w] = g$, $u(0) = 1$, $u'(0) = 2$, $v(0) = -1$, $v'(0) = -1$, $w(0) = 2$ and $w'(0) = -3$. Find the solution y of the initial value problem

$$L[y] = g, \quad y(0) = 0, \quad y'(0) = 1.$$

Problem 171. Find the characteristic polynomial, characteristic roots and the general (real) solution:

Part (A): $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0$

Part (B): $\dfrac{d^2y}{dx^2} - 4y = 0$

Part (C): $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 12y = 0$

Part (D): $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

Part (E): $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0$

Problem 172. Find the indicial polynomial, indicial roots and the general (real) solution:

Part (A): $x^2\dfrac{d^2y}{dx^2} - 2y = 0$

Part (B): $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = 0$

Part (C): $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 13y = 0.$

Problem 173. Find the general solution (in real form) for the ordinary differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \mathrm{e}^x \sec x.$$

Problem 174. The method of undetermined coefficients gives us the form of a particular solution (with parameters to be determined). For each of the following problems find the form of the particular (real) solution as given by the method of undetermined coefficients. Do NOT determine the coefficients.

Part (A): $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = x\mathrm{e}^{-x} + x\mathrm{e}^{x} + x\mathrm{e}^{3x}$

Part (B): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 20y = \mathrm{e}^{-2x}\cos(4x) + \mathrm{e}^{2x}\cos(4x) - \mathrm{e}^{-2x}$

Problem 175. Given

$$\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25 = \left(\lambda^2 - 4\lambda + 5\right)^2$$

find the general (real) solution of the ordinary differential equation

$$\frac{d^4y}{dx^4} - 8\frac{d^3y}{dx^3} + 26\frac{d^2y}{dx^2} - 40\frac{dy}{dx} + 25y = 0.$$

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6 Final Exam

Problem 176. Solve the initial value problem

$$\frac{dy}{dx} = 1 + \frac{4xy - y^2}{4x^2}, \quad y(1) = 2.$$

Problem 177. Find the characteristic polynomial, characteristic roots and the general (real) solution:

Part (A): $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 8y = 0$

Part (B): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Part (C): $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

Part (D): $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

Problem 178. For the following problem find the form of the particular (real) solution as given by the method of undetermined coefficients. Do NOT determine the coefficients.

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = x^2\mathrm{e}^{-3x}\cos(2x) + 2x\mathrm{e}^{-3x}.$$

Problem 179. The linear ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 2x\log(x), \quad x > 0,$$

has complementary function $y_c = c_1x + c_2x^2$. Use variation of parameters to find a particular solution. Then find the general solution.

Problem 180. A brine solution of 0.20 kg/L salt flows into a 200 L tank at the rate of 4.0 L/min. The well-mixed solution is drawn off at the rate of 2.0 L/min. The tank initially held 100 L of brine of concentration 0.15 kg/L salt. Find the concentration of brine in the tank at the moment of overflow.

Problem 181. (For this problem assume that a mass of 1 slug weighs exactly 32 lb. at the location where the problem is set.) A spring is hung from a support and a mass of $1/4$ slug is suspended from the spring. The spring stretches $8/13$ ft. as a result.

Part (A): Find the spring constant.

Part (B): If air resistance and other damping effects amount to $-4\frac{dx}{dt}$, where x is the downward displacement of the mass from equilibrium, find the equation of motion of the mass.

Part (C): If the mass is pulled down $1/3$ ft and then released (from rest) find the subsequent motion of the mass.

Problem 182. Find the inverse LAPLACE transforms

Part (A): $\mathcal{L}^{-1}\left\{\dfrac{s-10}{s^2+s-2}\right\}$

Part (B): $\mathcal{L}^{-1}\left\{\dfrac{2s^2+s+9}{(s+1)(s^2+4s+13)}\right\}$

Problem 183. Find the LAPLACE transform $X(s)$ of the solution $x(t)$ to the initial value problem

$$3\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 4x = \mathrm{e}^{2t}, \quad x(0) = -2, \quad x'(0) = 3.$$

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7 Contact Information

The contact information below is accurate as of Oct 8, 2000.

Bent E. Petersen
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605

phone numbers
office (541) 737-5163
home (541) 753-1829
fax (541) 737-0517

`bent@alum.mit.edu`
`petersen@math.orst.edu`
`http://ucs.orst.edu/~peterseb`
`http://www.peak.org/~petersen`
`http://web.orst.edu/~peterseb`