# Mth 256 Exam | Name: ID:

Bent Petersen `256w2001_exam.tex` March 19, 2001 Time: 110 minutes.

**Instructions:** $\Longrightarrow$
If you do not read the instructions, then how will you know what to do? Read them now.

Be sure to write your name in the space above.

- *You may use one* <u>*note-sheet*</u> *prepared in advance. You must put your name on your note-sheet, but* <u>*do not*</u> *turn in your note-sheet. Your note-sheet must be letter size,* $8.5 \times 11$ *inches, or A4 paper,* $21 \times 29.7$ *cm, or smaller. You may write on both sides of your note-sheet.*
- *Note-sheets may not be shared. If you do not bring a note-sheet you will have to do without any help notes.*
- *You may not use any books, notebooks nor additional note-sheets.*
- *You may use a calculator. Calculators and other equipment may not be shared.*
- <u>*For work-out problems*</u> *sketch your work neatly. Highlight your answer by drawing a frame around it. Scratch out irrelevant or incorrect work so it will be clear what you are submitting as a solution. Give exact answers when possible. Simplify your answer when reasonable to do so. Partial credit will be assigned only for relevant, clear, correct, legible work. If you do not show some relevant work or explain your solution, your grade may be 0.*
- <u>*For multiple-choice problems*</u> *indicate your choice in the answer box provided. You need not show any work nor offer any explanations for your answer. If you need to do some work, you may do it in the space provided, if any, or on the back of the examination sheets, but your work will not be graded.* ***You will be graded only on the letter you select and put in the provided answer box.*** <u>*Note this test does not use a scantron.*</u>
- *Use the backs of the examination sheets for scratch work.*

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*Please note* $\log(x)$ *means the natural logarithm of* $x$ *on this test.*

**Problem 1.** *(75 points).* Find the Laplace transform $Y(s)$ of the solution to the initial value problem

$$3\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = \cos(2t), \quad y(0) = 0, \quad y'(0) = 1.$$

**Problem 2.1.** *(15 points if correct, 0 points if wrong).* Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^3+s}\right\}.$$

**A.)** $-\frac{5}{3}e^{-t}+\frac{5}{3}e^{t/2}\cos\left(\frac{1}{2}\sqrt{3}t\right)-\frac{1}{3}e^{t/2}\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}t\right)$

**B.)** $2\sin(t)-3$

**C.)** $-3+3\cos(t)+2\cos(t)$

**D.)** $2\cos(t)-3\sin(t)$

**E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.1.*

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**Problem 2.2.** *(15 points if correct, 0 points if wrong).* Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{3s^2+2s-1}{(s+1)(s+2)(s-3)}\right\}.$$

**A.)** $\frac{1}{4}e^{-t}+e^{-2t}+\frac{7}{4}e^{3t}$

**B.)** $\frac{8}{5}e^{3t}+\frac{7}{5}e^{-2t}$

**C.)** $\frac{1}{4}e^{-t}+\frac{7}{4}e^{3t}$

**D.)** $\frac{1}{4}e^{-t}+\frac{7}{5}e^{-2t}+\frac{8}{5}e^{3t}$

**E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.2.*

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**Problem 2.3.** *(15 points if correct, 0 points if wrong).* Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^2 s}\right\}.$$

**A.)** $2te^{-t}+e^{-t}-1$ **B.)** $\left(2e^{-t}-1\right)^2$

**C.)** $e^{-t}\left(2e^{-t}-1\right)$ **D.)** $-2te^{-t}-3e^{-t}-t+3$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.3.*

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**Problem 2.4.** *(15 points if correct, 0 points if wrong).* Compute the Laplace transform

$$\mathcal{L}\left\{\cos(t)\sin(t)\right\}.$$

**A.)** $\frac{s^2}{(s^2+1)^2}$ **B.)** $\frac{s^2+2}{(s^2+4)s}$

**C.)** $\frac{1}{s^2+4}$ **D.)** $\frac{1}{(s^2+4)^2}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.4.*

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**Problem 2.5.** *(15 points if correct, 0 points if wrong).* Compute the Laplace transform

$$\mathcal{L}\left\{u(t-\pi)\cos(t)\right\}.$$

**A.)** $\frac{e^{-\pi s/2}s}{s^2+1}$ **B.)** $\frac{e^{-\pi s}s}{s^2+1}$

**C.)** $-\frac{e^{-\pi s/2}s}{s^2+1}$ **D.)** $-\frac{e^{-\pi s}s}{s^2+1}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.5.*

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**Problem 3.1.** *(15 points if correct, -3 points if wrong).* The method of undetermined coefficients gives a particular solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \left(t\,\mathrm{e}^t + 2\mathrm{e}^{-t}\right)\cos(2t)$$

of the form

**A.)** $(At+B)\mathrm{e}^t\cos(2t) + Ct\mathrm{e}^{-t}\cos(2t)$

**B.)** $t(At+B)\mathrm{e}^t\cos(2t) + t(Ct+D)\mathrm{e}^t\sin(2t) + E\mathrm{e}^{-t}\cos(2t) + F\mathrm{e}^{-t}\sin(2t)$

**C.)** $(At+B)\mathrm{e}^t\cos(2t) + (Ct+D)\mathrm{e}^t\sin(2t) + Et\mathrm{e}^{-t}\cos(2t) + Ft\mathrm{e}^{-t}\sin(2t)$

**D.)** $At\mathrm{e}^t\cos(2t) + Bt\mathrm{e}^t\sin(2t) + Et\mathrm{e}^{-t}\cos(2t) + Ft\mathrm{e}^{-t}\sin(2t)$

**E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.1.*

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**Problem 3.2.** *(15 points if correct, -3 points if wrong).* The method of undetermined coefficients gives a particular solution of

$$\frac{d^2y}{dt^2} - 4y = t\,\mathrm{e}^{2t} - 3t\,\mathrm{e}^{-2t} + 2t\,\mathrm{e}^{3t}$$

of the form

**A.)** $(At+B)\mathrm{e}^{2t} + (Ct+D)\mathrm{e}^{-2t} + (Et+F)\mathrm{e}^{3t}$

**B.)** $t(At+B)\mathrm{e}^{2t} + t(Ct+D)\mathrm{e}^{-2t} + (Et+F)\mathrm{e}^{3t}$

**C.)** $At\mathrm{e}^{2t} + Bt\mathrm{e}^{-2t} + (Ct+D)\mathrm{e}^{3t}$

**D.)** $At\mathrm{e}^{2t} + Bt\mathrm{e}^{-2t} + Ct\mathrm{e}^{3t}$

**E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.2.*

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**Problem 3.3.** *(15 points if correct, -3 points if wrong).* If $f(t) = \sin(t)$ and $g(t) = \mathrm{e}^t$ compute the convolution $f * g(t)$.

**A.)** $\frac{1}{2}\mathrm{e}^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)$ **B.)** $\frac{1}{2}\mathrm{e}^t + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$

**C.)** $\mathrm{e}^t\sin(t)$ **D.)** $\mathrm{e}^t\cos(t)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.3.*

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**Problem 3.4.** *(15 points if correct, -5 points if wrong).* Given

$$\mathcal{L}\left\{\sin\left(\sqrt{t}\right)\right\} = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/4s}s^{-3/2} \quad \text{and} \quad \int_0^t \sin\left(\sqrt{r}\right)\,\mathrm{d}r = 2\sin\left(\sqrt{t}\right) - 2\sqrt{t}\cos\left(\sqrt{t}\right)$$

compute the Laplace transform $\mathcal{L}\left\{2\sin\left(\sqrt{t}\right) - 2\sqrt{t}\cos\left(\sqrt{t}\right)\right\}$.

**A.)** $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/4s}s^{-1/2}$ **B.)** $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/4s}s^{-3/2}$

**C.)** $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/4s}s^{-5/2}$ **D.)** $\int_0^s \frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/4u}u^{-3/2}\,\mathrm{d}u$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.4.*

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**Problem 3.5.** *(15 points if correct, -5 points if wrong).* Solve the initial value problem

$$\tan(t)\frac{d^2y}{dt^2} + \frac{\sec(t)}{1+\tan(t)}\frac{dy}{dt} - 2\sec(2t)y = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

**A.)** $y(t) = \tan(t) - t$ **B.)** $y(t) = \sec(t) - 1$

**C.)** $y(t) = \tan(t) - t\sec(t)$ **D.)** $y(t) = \tan(t) - t\sec(2t)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.5.*

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**Problem 4.** *(75 points).* The linear homogeneous ordinary differential equation

$$t^2 \frac{d^2y}{dt^2} - t(t+2)\frac{dy}{dt} + (t+2)\,y = 0$$

has the general solution $y(t) = c_1\,t + c_2\,t\,\mathrm{e}^t$, $\quad (t > 0)$. Use variation of parameters to find the solution of the the linear inhomogeneous initial value problem

$$t^2 \frac{d^2y}{dt^2} - t(t+2)\frac{dy}{dt} + (t+2)\,y = 2t^4\,\mathrm{e}^t, \quad y(1) = 2, \quad y'(1) = 2.$$

## Some Laplace transforms

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\cos\omega t\}(s) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\}(s) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\cos\omega t\}(s) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\sin\omega t\}(s) = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\cosh\nu t\}(s) = \frac{s-a}{(s-a)^2-\nu^2}$$

$$\mathcal{L}\{e^{at}\sinh\nu t\}(s) = \frac{\nu}{(s-a)^2-\nu^2}$$

$$\mathcal{L}\left\{\sqrt{t}\right\}(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s} \quad (u = \text{ unit step})$$

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as} \quad (\delta = \text{ Dirac delta})$$

If $f$ is periodic with period $T > 0$ then $\mathcal{L}\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$.

$$\mathcal{L}\left\{1+\sum_{k=1}^{\infty}(-1)^k u(t-k)\right\}(s) = \frac{1}{s\,(1+e^{-s})}$$

$$\mathcal{L}\{|\sin(t)|\}(s) = \frac{\coth\left(\frac{\pi s}{2}\right)}{1+s^2}$$

## Some Laplace exchange formulæ

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\begin{aligned}
\mathcal{L}\left\{e^{at}f(t)\right\}(s) &= F(s-a) \\
\mathcal{L}\left\{t^n f(t)\right\}(s) &= (-1)^n F^{(n)}(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) &= \int_s^\infty F(r)\ \mathrm{d}r \quad \left(\text{if } \frac{f(t)}{t} \text{ integrable at } 0\right) \\
\mathcal{L}\left\{\int_0^t f(r)\ \mathrm{d}r\right\}(s) &= \frac{F(s)}{s} \\
\mathcal{L}\left\{\frac{df}{dt}\right\}(s) &= sF(s) - f(0) \quad (\text{if } f \text{ cont. on } [0,\infty)) \\
\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\}(s) &= s^2F(s) - sf(0) - f'(0) \quad (\text{if } f,\ f' \text{ cont. on } [0,\infty)) \\
\mathcal{L}\left\{u(t-a)f(t-a)\right\}(s) &= e^{-as}F(s) \quad (u = \text{ unit step}) \\
\mathcal{L}\left\{f(at)\right\}(s) &= \frac{1}{a}F\left(\frac{s}{a}\right).
\end{aligned}$$

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$ then $\mathcal{L}\{(f*g)(t)\}(s) = F(s)G(s)$ where $f*g$ is defined by $(f*g)(t) = \int_0^t f(t-r)g(r)\ \mathrm{d}r$.

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*Enjoy your Spring break!*

Please do not write in the boxes to the right. They are for your grades.
Do not be concerned if there are more boxes than problems.

| | Letter Grade |
|---|---|
| | ☐ *This test only* <br> ☐ *Cummulative* |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |

**Note:** There are 4 problems for a total of 300 points.