Mth 256 Exam | **Name:** **ID:**

Bent Petersen `256w2004-exam.tex` March 15, 2004 Time: 110 minutes.

Instructions: $\Longrightarrow$
If you do not read the instructions, then how will you know what to do? Read them now.

Be sure to write your name in the space above.

- *You may use one* 8.5×11 *inch note sheet prepared in advance. You may write on both sides of your note sheet.*
- *Note sheets may not be shared. If you do not bring a note sheet you will have to do without any help notes.*
- *You may not use any books, notebooks, additional note sheets nor note cards.*
- *You are expected to have a simple scientific calculator available for use on this test. Calculators and other equipment may not be shared.*
- *For multiple-choice problems place the letter corresponding to your answer in the box provided.*
- *Note that* $\log(x)$ *means the* natural logarithm *of* x.

Don't get nervous! The Laplace transform tables below are much more extensive than what you will need on this test.

Some Laplace exchange formulæ

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\begin{aligned}
\mathcal{L}\left\{\mathrm{e}^{at} f(t)\right\}(s) &= F(s-a) \\
\mathcal{L}\left\{t^n f(t)\right\}(s) &= (-1)^n F^{(n)}(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) &= \int_s^\infty F(r)\ \mathrm{d}r \quad \left(\text{if } \frac{f(t)}{t} \text{ integrable at } 0\right) \\
\mathcal{L}\left\{\int_0^t f(r)\ \mathrm{d}r\right\}(s) &= \frac{F(s)}{s} \\
\mathcal{L}\left\{\frac{df}{dt}\right\}(s) &= sF(s) - f(0) \quad (\text{if } f \text{ cont. on } [0,\infty)) \\
\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\}(s) &= s^2F(s) - sf(0) - f'(0) \quad \left(\text{if } f,\ f' \text{ cont. on } [0,\infty)\right) \\
\mathcal{L}\left\{u(t-a)f(t-a)\right\}(s) &= \mathrm{e}^{-as}F(s) \quad (u = \text{ unit step or Heaviside function}) \\
\mathcal{L}\left\{f(at)\right\}(s) &= \frac{1}{a}F\left(\frac{s}{a}\right).
\end{aligned}$$

If $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{g(t)\}(s) = G(s)$ then $\mathcal{L}\{(f*g)(t)\}(s) = F(s)G(s)$ where $f*g$ is defined by $(f*g)(t) = \int_0^t f(t-r)g(r)\ \mathrm{d}r$.

Some Laplace transforms

$$
\begin{aligned}
\mathcal{L}\{1\}(s) &= \frac{1}{s} \\
\mathcal{L}\left\{\mathrm{e}^{at}\right\}(s) &= \frac{1}{s-a} \\
\mathcal{L}\{t^n\}(s) &= \frac{n!}{s^{n+1}} \\
\mathcal{L}\{\cos\omega t\}(s) &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}\{\sin\omega t\}(s) &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}\left\{\mathrm{e}^{at}\cos\omega t\right\}(s) &= \frac{s-a}{(s-a)^2+\omega^2} \\
\mathcal{L}\left\{\mathrm{e}^{at}\sin\omega t\right\}(s) &= \frac{\omega}{(s-a)^2+\omega^2} \\
\mathcal{L}\left\{\mathrm{e}^{at}\cosh\nu t\right\}(s) &= \frac{s-a}{(s-a)^2-\nu^2} \\
\mathcal{L}\left\{\mathrm{e}^{at}\sinh\nu t\right\}(s) &= \frac{\nu}{(s-a)^2-\nu^2} \\
\mathcal{L}\left\{\sqrt{t}\right\}(s) &= \frac{\sqrt{\pi}}{2s^{3/2}} \\
\mathcal{L}\left\{t^n\mathrm{e}^{at}\right\}(s) &= \frac{n!}{(s-a)^{n+1}} \\
\mathcal{L}\{u(t-a)\}(s) &= \frac{\mathrm{e}^{-as}}{s} \quad (u = \text{ unit step or Heaviside function}) \\
\mathcal{L}\{\delta(t-a)\}(s) &= \mathrm{e}^{-as} \quad (\delta = \text{ Dirac delta})
\end{aligned}
$$

If f is periodic with period $T>0$ then $\mathcal{L}\{f(t)\} = \dfrac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1-\mathrm{e}^{-sT}}$.

$$
\begin{aligned}
\mathcal{L}\left\{1+\sum_{k=1}^{\infty}(-1)^k u(t-k)\right\}(s) &= \frac{1}{s\left(1+\mathrm{e}^{-s}\right)} \\
\mathcal{L}\{|\sin(t)|\}(s) &= \frac{\coth\left(\frac{\pi s}{2}\right)}{1+s^2}
\end{aligned}
$$

Problem 1. *(15 points if correct, -1 points if wrong).* Find the Laplace transform of

$$\frac{3}{2}\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-t}.$$

A.) $\frac{1}{2}\frac{2+s}{s^2-1}$ **B.)** $\frac{2+s}{s^2+1}$
C.) $\frac{2+s}{s^2-1}$ **D.)** $\frac{1}{2}\frac{2+s}{s^2+1}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 1.*

Problem 2. *(15 points if correct, -1 points if wrong).* Find the Laplace transform of

$$\cos(t) + 2\sin(t)$$

A.) $\frac{1}{2}\frac{2+s}{s^2-1}$ **B.)** $\frac{2+s}{s^2+1}$
C.) $\frac{2+s}{s^2-1}$ **D.)** $\frac{1}{2}\frac{2+s}{s^2+1}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.*

Problem 3. *(15 points if correct, -1 points if wrong).* Find the inverse Laplace transform of

$$\frac{2s+3}{s(s^2+1)}.$$

A.) $3 - 3\cos(t) + 2\sin(t)$ **B.)** $3\cos(t) + 2\sin(t)$
C.) $-3 + \frac{5}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t}$ **D.)** $\frac{5}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 3.*

Problem 4. *(15 points if correct, -1 points if wrong).* Find the inverse Laplace transform of

$$\frac{2s+3}{s(s^2-1)}.$$

A.) $3 - 3\cos(t) + 2\sin(t)$ **B.)** $3\cos(t) + 2\sin(t)$
C.) $-3 + \frac{5}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t}$ **D.)** $\frac{5}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.*

Problem 5. *(15 points if correct, -1 points if wrong).* Given the Laplace transform

$$\mathcal{L}\left\{\sin\left(\sqrt{t}\right)\right\} = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/(4s)}\, s^{-3/2}$$

find the Laplace transform

$$\mathcal{L}\left\{\mathrm{e}^{t}\sin\left(\sqrt{t}\right)\right\}$$

A.) $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-s-1/(4s)}\, s^{-3/2}$ **B.)** $\frac{\sqrt{\pi}}{2}\mathrm{e}^{s-1/(4s)}\, s^{-3/2}$
C.) $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/(4(s+1))}\,(s+1)^{-3/2}$ **D.)** $\frac{\sqrt{\pi}}{2}\mathrm{e}^{-1/(4(s-1))}\,(s-1)^{-3/2}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 5.*

Problem 6. *(15 points if correct, -1 points if wrong).* Consider the initial value problem

$$\frac{d^2y}{dt^2} + 4y = \sin(2t), \qquad y(0) = 0, \;\; y'(0) = 1.$$

Find the Laplace transform $Y(s)$ of the solution $y(t)$.

A.) $1/(s^2+4)^2$ **B.)** $(s+6)/(s^2+4)^2$
C.) $(s^2+6)/(s^2+4)^2$ **D.)** $(s+6)/(s^2+4)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 6.*

Problem 7. *(15 points if correct, -1 points if wrong).* Consider the initial value problem

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = \exp(3t), \quad y(0) = 1, \;\; y'(0) = 0$$

Find the Laplace transform $Y(s)$ of the solution $y(t)$.

A.) $\frac{s^2-2s-2}{(s-2)(s-3)(s+3)}$ **B.)** $\frac{s^2-2s-2}{(s-2)(s-3)^2}$
C.) $\frac{s^2+2s-2}{(s-2)(s+3)^2}$ **D.)** $\frac{s^2+2s-2}{(s-2)(s-3)(s+3)}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 7.*

Problem 8. *(15 points if correct, -1 points if wrong).* Find the Laplace transform of

$$u\left(t - \frac{\pi}{2}\right)\sin(t)$$

where u is the unit step function (Heaviside function).

A.) $\mathrm{e}^{-s\pi/2}\frac{s}{s^2+1}$ **B.)** $\mathrm{e}^{-s\pi/2}\frac{1}{s^2+1}$
C.) $\mathrm{e}^{-s\pi/2}\frac{s+1}{s^2+1}$ **D.)** $\mathrm{e}^{-s\pi/2}\frac{s-1}{s^2+1}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 8.*

Problem 9. *(15 points if correct, -1 points if wrong).* Find the Laplace transform of

$$f(t) = \begin{cases} 0 \text{ if } t \le 0 \\ t \text{ if } 0 < t \le 1 \\ 2 - t \text{ if } 1 < t \le 2 \\ 0 \text{ if } 2 < t \end{cases}$$

A.) $\frac{1+\mathrm{e}^{-s}}{s}$ **B.)** $\left(\frac{1+\mathrm{e}^{-s}}{s}\right)^2$
C.) $\frac{1-\mathrm{e}^{-s}}{s}$ **D.)** $\left(\frac{1-\mathrm{e}^{-s}}{s}\right)^2$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 9.*

Problem 10. *(15 points if correct, -1 points if wrong).* If we substitute $y = w\,\mathrm{e}^{\cos(t)}$ and then $v = w'$ (where the prime indicates differentiation with respect to t) in the differential equation

$$y'' + \sin(t)\,y' + \cos(t)\,y = 0$$

what differential equation for v do we obtain?

A.) $v' - \cos(x)\,v = 0$ **B.)** $v' + \cos(x)\,v = 0$
C.) $v' - \sin(x)\,v = 0$ **D.)** $v' + \sin(x)\,v = 0$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 10.*

Problem 11. *(15 points if correct, -1 points if wrong).* Compute the convolution product

$$\sin(t) * \cos(2t).$$

A.) $\sin(t) - \cos(2t)$ **B.)** $\cos(t) - \sin(2t)$
C.) $\sin(t) - \sin(2t)$ **D.)** $\cos(t) - \cos(2t)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 11.*

Problem 12. *(15 points if correct, -1 points if wrong).* Find the Laplace transform of the solution of the initial value problem

$$\begin{aligned} \frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2\,y &= \begin{cases} 1 \text{ if } 0 < t < 1 \\ 0 \text{ elsewise} \end{cases} \\ y(0) = 0,\ \ y'(0) &= 0. \end{aligned}$$

A.) $1/\left(s(s-1)(s-2)\right)$ **B.)** $\left(\mathrm{e}^{-2s} - \mathrm{e}^{-s}\right)/\left(s(s-1)(s-2)\right)$
C.) $\mathrm{e}^{-2s} - \mathrm{e}^{-s}$ **D.)** $\left(\mathrm{e}^{-2s} + \mathrm{e}^{-s}\right)/\left(s(s-1)(s-2)\right)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 12.*

Use this page and the backs of all the pages for scratch work. Do not write in the boxes below. Enjoy your Spring break!

1 (15)	2 (15)	3 (15)	4 (15)	5 (15)	6 (15)	7 (15)	8 (15)
9 (15)	10 (15)	11 (15)	12 (15)				

12 problems 180 points	TOTAL: