Applied Differential Equations

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Contents

This list of problems will be augmented during the quarter. Some of the problems are from old lists of sample problems, old tests or old assignments. Thus there may be some duplication, or at least, near duplication. Nonetheless, you may find my list, from Spring 1997 (see my web page), of 117 sample problems for first order ordinary differential equation, as well as numerous other problems, useful as well.

1 Linear First Order Ordinary Differential Equations

Problem 1.1. Solve the initial value problem

$$\frac{dy}{dt} + \frac{1}{t}y = t^2, \qquad y(1) = 1.$$

Then choose the number in the list below closest to $y(2)$.

A.) 1.000 **B.)** 2.375
C.) 3.000 **D.)** 3.125 **E.)** 3.250

☐ ←*Letter corresponding to your answer to problem 1.1.*

Problem 1.2. Solve the initial value problem

$$t^5\frac{dy}{dt} - t^4 y = t, \qquad y(1) = -1.$$

Then choose the number in the list below closest to $y(2)$.

A.) -1.200 **B.)** -1.523
C.) -1.531 **D.)** -1.662 **E.)** -2.132

☐ *←Letter corresponding to your answer to problem 1.2.*

Problem 1.3. Solve the initial value problem

$$\frac{dy}{dt} + \tan(t)y = \cos^2(t), \qquad y(\frac{\pi}{4}) = 2$$

and then compute $y(\frac{\pi}{3})$.

A.) 0 **B.)** 1
C.) $\frac{1}{4}\sqrt{3} + \frac{3}{4}\sqrt{2}$ **D.)** $-\frac{1}{4}\sqrt{3} - \frac{3}{4}\sqrt{2}$ **E.)** None of the above.

☐ *←Letter corresponding to your answer to problem 1.3.*

Answers

1.1	2.375 B	1.2	-1.5312500 C	1.3	-1.5312500 C

2 NEWTON's Law of Cooling

NEWTON's law of cooling (or heating) states that when a body at temperature T is immersed in a medium at temperature A then the rate of change of T is proportional to the difference in the temperatures. Explicitly

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. In the case of an object of very small heat capacity, compared to the heat capacity of the surrounding medium, we may frequently assume that the ambient temperatute A is constant.

Note in some cases STEFAN's law of radiation

$$\frac{dT}{dt} = -k\left(T^4 - A^4\right)$$

is more appropriate. Since this equation is not translation invariant in T, nor in A, we suspect that we must use absolute temperature, as is in fact the case.

Fortunately in our simple examples NEWTON's law is adequate and we may therefore use any temperature scale. Out of habit, I usually use the FAHRENHEIT scale.

Problem 2.1. A cup of coffee at 194°F is brought into a classroom with constant temperature A°F (ambient). If T is the temperature of the coffee then according to NEWTON's law of cooling

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant depending on the insulating properties of the cup. Five minutes later the coffee has temperature $169°$F, and another 5 minutes later the temperature of the coffee is only $149°$F. Find the temperature of the room.

A.) $67°$F **B.)** $69°$F
C.) $71°$F **D.)** $73°$F **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 2.1.*

Problem 2.2. A cup of coffee at $194°$F is brought into a classroom with constant temperature $A°$F (ambient). If T is the temperature of the coffee then according to NEWTON's law of cooling

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant depending on the insulating properties of the cup. Five minutes later the coffee has temperature $169°$F, and another 5 minutes later the temperature of the coffee is only $149°$F. Find the temperature of the coffee an additional 10 minutes later. Choose the closest number from the list below.

A.) $114°$F **B.)** $116°$F
C.) $118°$F **D.)** $120°$F **E.)** $122°$F

☐ ←*Letter corresponding to your answer to problem 2.2.*

Problem 2.3. A cup of coffee initially at temperature $T_0 = 185°$ F is brought into a room at temperature $A = 68°$ F. The heat capacity of the room (compared to the coffee) is so large that we may regard A as being constant. After 8 minutes the temperature of the coffee is $145°$ F. What temperature will the coffee be an additional 3 minutes later? Choose the closest value from the list below.

A.) $120°$F **B.)** $122°$F
C.) $132°$F **D.)** $134°$F **E.)** $136°$F

☐ ←*Letter corresponding to your answer to problem 2.3.*

Problem 2.4. A thermometer reading $92°$ F is immersed in a cooler liquid. After 3 seconds the thermometer reads $80°$ F. Another 3 seconds later it reads $76°$ F. What is the temperature of the fluid? Choose the closest value from the list below.

A.) $73°$F **B.)** $74°$F
C.) $75°$F **D.)** $76°$F **E.)** $77°$F

☐ ←*Letter corresponding to your answer to problem 2.4.*

Problem 2.5. A thermometer is immersed in a cool liquid. After 2 seconds it reads $101.3°$ F. After another 3 seconds the thermometer reads $94.1°$ F. Another 7 seconds later it reads $89.8°$ F. What is the temperature of the fluid? Choose the closest value from the list below.

A.) $89.0°$F **B.)** $89.2°$F
C.) $89.3°$F **D.)** $89.6°$F **E.)** $89.7°$F

☐ ←*Letter corresponding to your answer to problem 2.5.*

Problem 2.6. Consider a cup of coffee in a room of temperature $A = 70^\circ$ F. The cup is sitting on a small heating pad which is supposed to keep the coffee warm. If T is the temperature of the coffee then

$$\frac{dT}{dt} = -k\,(T - A) + U$$

where U is a constant depending on the heater and the cup and k is a constant depending on the cup. Initially the temperature of the coffee is 183° F, 5 minutes later the temperature is 155° F, and an additional 5 minutes later the temperature is 135° F. Find the temperature T as a function of time. Compute $T(45)$. Choose the closest value from the list below. (You will probably conclude that you need a better heater!)

A.) $85.0°\mathrm{F}$ **B.)** $87.2°\mathrm{F}$
C.) $89.7°\mathrm{F}$ **D.)** $92.6°\mathrm{F}$ **E.)** $99.3°\mathrm{F}$

☐ *←Letter corresponding to your answer to problem 2.6.*

Answers

2.1	69 B	2.2	120.2 D	2.3	133.8195712 D
2.4	74 B	2.5	89.20773632 B	2.6	89.74322531 C

3 Mixing Problems

Consider a tank initially containing a volume V_0 of brine (salt dissolved in water) of concentration ϵ_0. Let $Q(t)$ be the amount of salt in the tank at time t. Note the initial amount of salt is given by $Q_0 = \epsilon_0 V_0$. Suppose brine of concentration ϵ_i flows into the tank at the volume rate r_i. Then salt is entering the tank at the rate $\epsilon_i r_i$. Suppose some evaporation takes place (pure water), say at the volume rate r_e. Suppose in addition to everything else the well-mixed brine solution is pumped out of the tank at the volume rate r_o. The concentration of this outflow is $\frac{Q}{V}$ where V is the current volume of brine in the tank.

We assume that r_i, r_e and r_o are constant. Then clearly

$$V = V_0 + (r_i - r_e - r_o)t$$

and

$$\frac{dQ}{dt} = \epsilon_i r_i - \frac{Q}{V} r_o, \qquad Q(0) = Q_0.$$

Problem 3.1. A 100 gallon tank initially contains 50 gallons of brine of concentration 1 oz salt per gallon. Brine of concentration 2 oz salt per gallon flows into the tank at 5 gallons per minute. The well-mixed solution is pumped out at 3 gallons per minute. Find the concentration of salt in the brine in the tank at the very moment of overflow. Select the closest number from the list below.

A.) 2.125000 **B.)** 1.994961
C.) 1.911274 **D.)** 1.823223 **E.)** 1.223417

☐ *←Letter corresponding to your answer to problem 3.1.*

Problem 3.2. A 100 gallon tank initially contains 50 gallons of brine of concentration 1 oz salt per gallon. Brine of concentration 2 oz salt per gallon flows into the tank at 5 gallons per minute. The well-mixed solution is pumped out at 2 gallons per minute. In addition water evaporates from the tank at the rate 1 gallon (unrealistic) per minute. Find the concentration of salt in the brine in the tank at the very moment of overflow. Select the closest number from the list below.

A.) 2.125000 **B.)** 1.994961
C.) 1.911274 **D.)** 1.823223 **E.)** 1.223417

☐ ←*Letter corresponding to your answer to problem 3.2.*

Answers

3.1	1.823223305 D	3.2	2.125000 A	3.3	

4 Separable Ordinary Differential Equation

In class we derived the solution of the separable ordinary differential equation

$$\frac{dy}{dt} = \frac{g(t)}{h(y)}$$

in the implicit form

$$\int h(y)\ \mathrm{d}y = C + \int g(t)\ \mathrm{d}t$$

Problem 4.1. Solve the ode

$$\frac{dy}{dt} = \frac{\tan(y)}{1+t^2}.$$

A.) $y = \arcsin\left(\mathrm{e}^{\arctan(t)}\right) + C$ **B.)** $y = \arcsin\left(\mathrm{e}^{\arctan(t)} + C\right)$
C.) $y = \arcsin\left(\mathrm{e}^{\arctan(t)}C\right)$ **D.)** $y = \arcsin\left(\mathrm{e}^{\arctan(t)}\right) C$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.1.*

Problem 4.2. Solve the ode

$$\frac{dy}{dt} = \frac{ty}{1+t^2}.$$

A.) $C + \sqrt{1+t^2}$ **B.)** $C\sqrt{1+t^2}$
C.) $C + \frac{1}{2}\left(1+t^2\right))$ **D.)** $C\left(1+t^2\right)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.2.*

Problem 4.3. Solve the initial value problem

$$\frac{dy}{dt} - y^2 = y, \quad y(0) = 1.$$

Then evaluate $\log\left(\sqrt{2}\right)$.

A.) 1 **B.)** $\frac{1}{\sqrt{2}-1}$
C.) $\sqrt{2}-1$ **D.)** $\frac{1}{\sqrt{2}}$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.3.*

Problem 4.4. Solve the initial value problem

$$\frac{dy}{dt} = t^2\left(1-y\right), \quad y(0) = 1.$$

A.) 1 **B.)** $1+\exp(-t^3/3)$
C.) $1-\exp(-t^3/3)$ **D.)** $1+\exp(-t^3)$ **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.4.*

Problem 4.5. Solve the initial value problem

$$\frac{dy}{dt} = ty - t^2 y, \quad y\left(\frac{3}{2}\right) = 1.$$

Then compute $y(0)$.

A.) 1 **B.)** 3
C.) 4 **D.)** -1 **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.5.*

Problem 4.6. Solve the initial value problem

$$\frac{dy}{dt} = \mathrm{e}^{2t-3y}, \quad y(0) = \frac{1}{3}\log(3).$$

Then compute $y(\log(2))$.

A.) $\log(3)$ **B.)** $\log(15/2)$
C.) $\log(30)/3$ **D.)** 1 **E.)** None of the above.

☐ ←*Letter corresponding to your answer to problem 4.6.*

Answers

4.1	C	4.2	B	4.3	B
4.4	A	4.5	A	4.6	$\frac{1}{3}\log\left(\frac{15}{2}\right)$ E

5 TORRICELLI–BORDA Principle

Consider a tank initially full of some fluid. Assume that the tank is drained through an orifice in the bottom. According to TORRICELLI (1608–1647) the fluid issues from the orifice with a velocity u given by $u = \sqrt{2gh}$, where $g = 32.2$ ft/sec^2 is the acceleration of gravity and h (the head) is the instantaneous height of the fluid surface above the orifice. Thus if α is the cross–sectional area of the stream of fluid issuing from the orifice then

$$\frac{dV}{dt} = -\alpha u$$

where V is the instantaneous volume of fluid in the tank. If $A(h)$ is the cross–sectional area of the tank at height h then from calculus we know that

$$\frac{dV}{dt} = A(h)\,\frac{dh}{dt}.$$

Combining the two facts above we obtain the following differential equation for h:

$$A(h)\,\frac{dh}{dt} = -\alpha\,\sqrt{2gh}.$$

According to BORDA (1733–1799), $\alpha = \beta a$ where a is area of the orifice and β is BORDA's constant. The constant β depends on the fluid. In most cases we have $\frac{1}{2} \leq \beta \leq 1$. For water we have $\beta = 0.6$ (approximately).

Answers

5.1		5.2		5.3	

6 Falling Bodies

Gravity

The Earth's acceleration of gravity at sea-level g varies with latitude. Here's a short table

equator	9.77989 m/sec^2	32.0862 ft/sec^2
45° latitude	9.80621 m/sec^2	32.1725 ft/sec^2
pole	9.83210 m/sec^2	32.2575 ft/sec^2
conventional values	9.80655 m/sec^2	32.1737 ft/sec^2
typical textbook values	9.80 – 9.81 m/sec^2	32.1 – 32.2 ft/sec^2

I will try to use $g = 9.81$ m/sec^2 or $g = 32.2$ ft/sec^2 consistently.

Water

One ft^3 pure water at sea level weighs about 62.3 lb. If it is sea water though then it weighs about 64.0 lb. One ft^3 is 7.48126 US gallons so a 55 gallon drum has a volume of 7.3517 ft^3. If the drum is submerged in water it will experience a bouyant force of about 458.0 lb in fresh water or 470.5 lb in sea water.

Falling in air

Consider a parachutist falling through the atmosphere with downward component of velocity v. If we assume the atmospheric drag is proportional to v^2 we obtain the equation of motion

$$m\frac{dv}{dt} = mg - kv^2$$

where m is the mass of the parachutist and equipment and g is the acceleration of gravity. Since our model applies near the Earth's surface we assume g is constant, say g = 32.2 ft/sec^2, and we formulate our model in terms of the weight $W = mg$ of the parachutist and equipment. We have

$$\frac{dv}{dt} = g - \frac{gk}{W}v^2.$$

Introducing the constants

$$\alpha = \sqrt{\frac{W}{k}} \text{ and } \beta = \frac{g}{\alpha}$$

we obtain

$$\frac{dv}{dt} = \frac{\beta}{\alpha}\left(\alpha^2 - v^2\right).$$

Answers

6.1		6.2		6.3	

7 Integrating Factors and Exact ODE

Problem 7.1. Identify an integrating factor for the ordinary differential equation

$$xy^2 + \left(1 + y^2\right)\frac{dy}{dx}.$$

A.) $\frac{1}{x^2}$ **B.)** $\frac{1}{y^2}$
C.) $\left(1 + y^2\right)$ **D.)** $xy^2 + y^2$ **E.)** None of the above.

☐ *←Letter corresponding to your answer to problem 7.1.*

Problem 7.2. Identify an integrating factor for the ordinary differential equation

$$(x-3)\cos(y) + (x+1)\sin(y)\frac{dy}{dx} = 0.$$

A.) $\mathrm{e}^{-x}(x+1)^3$ **B.)** $\mathrm{e}^{-x}(x+1)^3\sin(y)$
C.) $\mathrm{e}^{-x}(x+1)\sin(y)$ **D.)** $(x+1)^3\sin(y)$ **E.)** None of the above.

☐ *←Letter corresponding to your answer to problem 7.2.*

Problem 7.3. Identify an integrating factor for the ordinary differential equation

$$12 + 5xy + \left(6\frac{x}{y} + 3x^2\right)\frac{dy}{dx} = 0.$$

A.) x^2y **B.)** $x^{-2}y^{-1}$
C.) $x\,(3 + xy)$ **D.)** $x^{-1}(3 + xy)^{-1}$ **E.)** None of the above.

☐ *←Letter corresponding to your answer to problem 7.3.*

Problem 7.4. Find an integrating factor of the form $\mu = x^m y^n$ for the ordinary differential equation

$$12 + 9y - 4xy^3 - 3x(-1 + xy^2)\frac{dy}{dx} = 0.$$

A.) $m = -1$, $n = 3$ **B.)** $m = 3$, $n = 1$
C.) $m = 2$, $n = 2$ **D.)** $m = 2$, $n = 0$ **E.)** None of the above.

☐ *←Letter corresponding to your answer to problem 7.4.*

Answers

7.1	B	7.2	A	7.3	D
7.4	D	7.5		7.6	

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