Applied Differential Equations – Mth 256

Archive – Winter 1993 Files

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This archive contains the sample problems and tests from Mth 256 Winter 1993. The original test instructions, headers and formatting have not been preserved.

Contents

1 Sample Problems

Problem 1. A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

Problem 2. Find the family of orthogonal trajectories to the one-parameter family of cubics

$$y = \alpha x^3, \qquad \alpha = \text{ arbitrary constant.}$$

Problem 3. Find an integrating factor which depends only on y and then solve the differential equation

$$(2y + y^2 - 6xy)\,dx \;+\; (4x + 3xy - 6x^2)\,dy = 0.$$

Problem 4. Find the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-2s-3}\right\}.$$

Problem 5. Solve the ordinary differential equation

$$y\frac{dy}{dx} = x\,e^{x^2+y^2}.$$

Problem 6. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}, \qquad y(1) = 2.$$

Problem 7. Solve the initial value problem

$$\frac{dy}{dx} + y\cos(x) = \cos(x), \qquad y(0) = 3.$$

Problem 8. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{\big(x+y-1\big)\big(x+y\big)}{2x+2y+1}$$

Problem 9. Solve the initial value problem

$$\big(3x^2y^2 - 2xy^3 - 2x - 1\big)\,dx + \big(2x^3y - 3x^2y^2 - 8y^3 - y + 1\big)\,dy = 0, \qquad y(0) = 2.$$

Problem 10. A tank initially contains 100 L of brine of concentration 0.6 g/L salt. Brine of concentration 2.1 g/L runs into the tank at 6.0 L/min and the well-mixed solution is drained off at 4.0 L/min. Find the concentration of salt in the tank at the moment that the tank contains 220 L brine.

Problem 11. Consider the 1-parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1 \qquad \alpha \text{ a constant (the parameter).}$$

Find the 1-parameter family of orthogonal trajectories.

Problem 12. A thermometer is brought into a certain room. The room has temperature $A = 25°$ C. If T is the temperature displayed by the thermometer then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads 21.4° C. An additional 20 seconds later it reads 23.4° C. What was the initial reading on the thermometer at the time that it was first brought into the room?

Problem 13. Use the substitution $y = x^2 w$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{2y^2 + x^3}{xy}.$$

Problem 14. Compute and simplify $\mathcal{L}\left\{e^{2t}\cos 3t\right\}$.

Problem 15. Compute and simplify $\mathcal{L}\left\{\cos^2 2t\right\}$.

Problem 16. Compute and simplify $\mathcal{L}\left\{e^t \cos t \sin t\right\}$.

Problem 17. Compute and simplify $\mathcal{L}\left\{\frac{e^t - e^{-t}}{t}\right\}$.

Problem 18. Compute and simplify $\mathcal{L}\left\{t \sin 2t\right\}$.

Problem 19. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\}.$$

Problem 20. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{(s+1)^2(s-1)}\right\}.$$

Problem 21. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s(s^2+4)}\right\}.$$

Problem 22. Find the LAPLACE transform of the solution to the initial value problem

$$2y'' - 3y' + 2y = te^t, \qquad y(0) = -1, \quad y'(0) = 2.$$

Problem 23. If

$$y(t) + \int_0^t y(r)\, dr = 1$$

use the laplace transform to find y(t).

2 Test 1

Problem 24. Find the general solution for the following ordinary differential equations.

(A) $x\,\dfrac{dy}{dx} = y\,\log|y|$ (B) $xy\,\dfrac{dy}{dx} = 2y^2 + x^2.$

Problem 25. Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} - (\tan x)y = \sec x.$$

Problem 26. Solve the initial value problem

$$\frac{dy}{dx} = 2 - (2x - y)^2, \qquad y(0) = 1.$$

Hint: The substitution $z = 2x - y$ is illuminating.

Problem 27. Solve the initial value problem

$$(4x + 4y + 3)dx + (4x - 6y - 2)dy = 0, \qquad y(2) = 1.$$

Problem 28. For what values of m and n is $x^m y^n$ an integrating factor for the differential equation

$$(3y + 12xy^2)dx + (4x + 15x^2 y)dy = 0.$$

Problem 29. A 200 L tank initially contains 100 L of brine of concentration 3 g/L salt (i.e., 3 grams salt per liter water). Brine of concentration 5 g/L salt runs into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the solution in the tank at the moment that the tank begins to overflow.

Here's a few fact concerning the LAPLACE transform you may find useful in this problem and the next one.

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$.
If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.
If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$ if f is continuous.

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}.$$

Problem 30. Consider the initial value problem

$$3\,\frac{dy}{dx} - 2y = e^t \sin(2t) \qquad y(0) = -1.$$

Find the LAPLACE transform $Y(s)$ of the solution $y(t)$ of the initial value problem. (It is not necessary to solve the initial value problem.)

Problem 31. Find the inverse LAPLACE transform of

$$Y(s) = \frac{3s^3 - 2s^2 + 6s + 1}{s^4 - 1}.$$

Hint: Expand in partial fractions.

3 Test 2

Problem 32. Find the general solution (in real form) for each of the following linear homogeneous ordinary differential equations.

(A) $y'' - 4y' = 0$

(B) $y'' - 4y' + 3y = 0$

(C) $y'' - 4y' + 4y = 0$

(D) $y'' - 4y' + 5y = 0$

(E) $y'' - 4y = 0$

(F) $y'' + 4y = 0$

Problem 33. Find the general solution (in real form) for each of the following linear homogeneous ordinary differential equations.

(A) $y'''' + 2y'' + y = 0$

(B) $y'''' - 2y'' + y = 0$

Problem 34. The second order homogenous linear ordinary differential equation

$$x^3y'' - xy' + y = 0$$

has the solution $y_1(x) = x$. Use reduction of order to find another solution y_2 such that $y = c_1y_1 + c_2y_2$ is a general solution.

Problem 35. Use the method of undetermined coefficients to find a particular solution of the ordinary differential equation

$$y'' - 3y' + 2y = x^2 - x.$$

Problem 36. Use the method of undetermined coefficients to find a particular solution of the ordinary differential equation

$$y'' + 2y' = x^2 - 3.$$

Problem 37. Use the method of undetermined coefficients to find a particular solution of the ordinary differential equation

$$y'' + 2y' + 5y = 4e^{-x}(x + 1)\cos(2x).$$

4 Final Exam

The final exam originally included a table of LAPLACE transforms. That table has been omitted here.

Problem 38. The BESSEL equation of order $\frac{1}{2}$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \frac{1}{4})y = 0$$

has fundamental solution set $\{y_1, y_2\}$ where

$$y_1(x) = x^{-1/2}\cos(x) \quad \text{and} \quad y_2(x) = x^{-1/2}\sin(x).$$

Find the solution y which satisfies the initial coditions

$$y(\pi) = 1 \quad \text{and} \quad y'(\pi) = 0$$

Problem 39. Find the family of orthogonal trajectories to the one-parameter family of ellipses

$$x^2 + 2y^2 = \alpha.$$

Here α is the parameter, that is, an arbitrary constant.

Problem 40. A 100 gal tank initially contains 20 gal of brine of concentration 0.36 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the concentration of salt in the tank at the very moment that it begins to overflow.

Problem 41. The CAUCHY–EULER differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 + 2x^2 - x.$$

has the *complementary solution* $c_1x + c_2x^2$. Use VARIATION OF PARAMETERS to find a *particular solution*.

Problem 42. Given that $y_1 = e^x$ is a solution of the ordinary differential equation

$$x\frac{d^2y}{dx^2} - (3x+1)\frac{dy}{dx} + (2x+1)y = 0$$

use REDUCTION OF ORDER to find a second solution y_2 such that $c_1y_1 + c_2y_2$ is a fundamental solution.

Problem 43. Find the general solution

(A) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$

(B) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0$

(C) $4\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 9y = 0$

(D) $6x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} - y = 0$

(E) $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 5y = 0$

(F) $4x^2\dfrac{d^2y}{dx^2} + y = 0.$

Problem 44. Find the general solution of the ordinary differential equation

$$y\frac{d^2y}{dx^2} + (2y^2+1)\left(\frac{dy}{dx}\right)^2 = 0.$$

Problem 45. Find the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s^2-1)(s+1)}\right\}.$$

Problem 46. Consider the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Find the LAPLACE transform of the solution to this initial value problem.

Problem 47. Use the METHOD OF UNDETERMINED COEFFICIENTS to find a particular solution

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (4x^3 - x)\,e^{3x}.$$

5 Contact Information

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