Applied Differential Equations – Mth 256

Archive – Winter 1994 Files

Jan 11, 2001

This archive contains the sample problems and tests from Mth 256 Winter 1994. The original test instructions, headers and formatting have not been preserved.

Contents

1 Sample Problem Set A

Problem 1. Find the general solution of the ordinary differential equations

$$\text{(A)}\quad xy\frac{dy}{dx} = x^2 + 1 \qquad\qquad \text{(B)}\quad xy\frac{dy}{dx} = x^2 + y^2.$$

Problem 2. Solve the initial value problem

$$(1+x^2)\,\frac{dy}{dx} + \left(3x + \frac{1}{x}\right) y = 6x + 2, \quad y(1) = 2.$$

Problem 3. Find the orthogonal trajectories for the family of ellipses

$$2y^2 + x^2 = \alpha \quad (\alpha \text{ is an arbitrary parameter}).$$

Problem 4. A large tank contains 80 gallons of brine of concentration 1.621 oz/gal salt. Brine of concentration 2.121 oz/gal salt flows into the tank at 3 gal/min. The well-mixed solution is drawn off at the rate of 4 gal/min. When will the brine in the tank reach a *concentration* of 2.013 oz/gal salt?

Problem 5. Find the general solution of the ordinary differential equation

$$(1 + \log(xy))\; dx + \left(1 + \frac{x}{y}\right)\; dy = 0.$$

Problem 6. A thermometer initially reading 62° F is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads 167° F. After an additional 1 second it reads 179° F. If A denotes the temperature of the coffee, T denotes the temperature reading of the thermometer and t denotes time in seconds then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. We regard the temperature A of the coffee also as constant. Find the temperature of the coffee.

Problem 7. A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

Problem 8. Find the family of orthogonal trajectories to the one-parameter family of cubics

$$y = \alpha x^3, \qquad \alpha = \text{ arbitrary constant.}$$

Problem 9. Find an integrating factor which depends only on y and then solve the differential equation

$$(2y + y^2 - 6xy)\, dx \;+\; (4x + 3xy - 6x^2)\, dy = 0.$$

2 Sample Problem Set B

Problem 10. Solve the ordinary differential equation

$$y\frac{dy}{dx} = x\,e^{x^2+y^2}.$$

Problem 11. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}, \qquad y(1) = 2.$$

Problem 12. Solve the initial value problem

$$\frac{dy}{dx} + y\cos(x) = \cos(x), \qquad y(0) = 3.$$

Problem 13. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{\bigl(x+y-1\bigr)\bigl(x+y\bigr)}{2x+2y+1}$$

Problem 14. Solve the initial value problem

$$\bigl(3x^2y^2 - 2xy^3 - 2x - 1\bigr)\,dx + \bigl(2x^3y - 3x^2y^2 - 8y^3 - y + 1\bigr)\,dy = 0, \qquad y(0) = 2.$$

Problem 15. A tank initially contains 100 L of brine of concentration 0.6 g/L salt. Brine of concentration 2.1 g/L runs into the tank at 6.0 L/min and the well-mixed solution is drained off at 4.0 L/min. Find the concentration of salt in the tank at the moment that the tank contains 220 L brine.

Problem 16. Consider the 1-parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1 \qquad\qquad \alpha \text{ a constant (the parameter).}$$

Find the 1-parameter family of orthogonal trajectories.

Problem 17. A thermometer is brought into a certain room. The room has temperature $A = 25^\circ$ C. If T is the temperature displayed by the thermometer then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads 21.4° C. An additional 20 seconds later it reads 23.4° C. What was the initial reading on the thermometer at the time that it was first brought into the room?

Problem 18. Use the substitution $y = x^2 w$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{2y^2 + x^3}{xy}.$$

3 Sample Problem Set C

Problem 19. Solve the initial value problem

$$\left(1+x^2\right)\,\frac{dy}{dx} = y\,x, \quad y(1) = 2.$$

Problem 20. Given the one–parameter family of curves

$$x^2 + y^2 = \alpha + 2\,\log(x) \qquad (\alpha \text{ is the parameter})$$

find the one–parameter family of orthogonal trajectories.

Problem 21. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing sinusoidally, say

$$A = A_0 + A_1\,\cos(\omega t)$$

where A_0, A_1 and $\omega > 0$ are constants, and t is time. According to Newton's law of cooling we have

$$\frac{dT}{dt} = -k\,(T - A)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1, ω and k. (Do not neglect the arbitrary constant.) Which part of your solution represents the steady–state? What is the amplitude, period and phase of the steady–state solution?

Problem 22. Solve the initial value problem

$$x^2\,\frac{dy}{dx} + 4\,y = 1, \qquad y(1) = 0.$$

Problem 23. Solve the initial value problem

$$x\,\frac{dy}{dx} + 2\,y\log x = 4\,\log x, \qquad y(1) = -1.$$

Problem 24. Solve the ordinary differential equation

$$\frac{dy}{dx} = (x+y)^2 + 2\,(x+y).$$

Problem 25. Solve the ordinary differential equation

$$\frac{dy}{dx} = (x+y)^2 + 3\,(x+y) + 1.$$

Problem 26. Solve the initial value problem

$$\frac{dp}{dt} = \mathrm{e}^{2t}\,p(1-p), \qquad p(0) = 0.5.$$

Find

$$\lim_{t\to\infty}\,p(t).$$

4 Test 1

Problem 27. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x}\,\log(x), \quad y(\mathrm{e}) = \mathrm{e}$$

where e is the base of the natural logarithm (Euler's number).

Problem 28. Solve the initial value problem

$$x\,\frac{dy}{dx} + 4\,y = x, \qquad y(1) = -1.$$

Problem 29. Use the substitution $w = 2\,x + 3\,y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(2x+3y)^2 - 4\,(2x+3y) + 4}{6\,(2x+3y)}.$$

Problem 30. The ordinary differential equation

$$\left(6\,xy + y^2\right)\,dx + \left(9\,x^2 - 6 + 4\,xy\right)\,dy = 0$$

has an integrating factor μ depending only on y. Find the integrating factor and then solve the ordinary differential equation.

Problem 31. A 200 gal tank initially contains 100 gal fresh water. Brine of concentration 1.2 oz/gal salt flows into the tank at the rate 3 gal/min. The well–mixed solution is drawn off at the rate 2 gal/min. Find the *concentration* of salt in the tank at the very moment that it begins to overflow.

Problem 32. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing linearly (for a while at least), say

$$A = A_0 + A_1\,t$$

where A_0 and A_1 are constants, and t is time. According to NEWTON's law of cooling we have

$$\frac{dT}{dt} = -k\,\left(T - A\right)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1 and k. (Do not neglect the arbitrary constant.)

5 Sample Problem Set D

Sample Problem Set D is the same as Test 1 and so has been omitted. It was originally distributed to be used for review since I was slow to grade the papers and to return them.

6 Sample Problem Set E

Problem 33. For what value of λ is $y = x^{\lambda}\log(x)$ a solution of the ordinary differential equation $x^2y'' - 5xy' + 9y = 0$?

Problem 34. Solve the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4\mathrm{e}^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Problem 35. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$ **(B)** $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0$ **(C)** $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

(D) $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = 0$ **(E)** $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 3y = 0$ **(F)** $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 5y = 0$

(G) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 0$ **(H)** $\dfrac{d^2y}{dx^2} + 4y = 0$ **(I)** $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0$

Problem 36. The ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{1/2}, \quad x > 0$$

has a particular solution of the form $Ax^{1/2}$ where A is a constant. **(A)** Find the constant A. **(B)** Does the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{-1}, \quad x > 0$$

have a solution of the form Ax^{-1}? Explain your answer.

Problem 37. Find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x\left(\mathrm{e}^{2x} + \mathrm{e}^{-2x}\right).$$

Problem 38. Consider the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(x^2 + 2\right)y = x^3\sec x, \quad x \geq 0.$$

Given that the complementary solution is

$$c_1 x\cos x + c_2 x\sin x$$

use variation of parameters to find the general solution.

7 Sample Problem Set F

Problem 39. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2\,y = 0 \qquad \textbf{(B)}\quad 4\,\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} - 3\,y = 0 \qquad \textbf{(C)}\quad 4\,\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} + y = 0$$

Problem 40. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad x^2\frac{d^2y}{dx^2} + 5x\,\frac{dy}{dx} + 13\,y = 0 \qquad \textbf{(B)}\quad 4\,x^2\frac{d^2y}{dx^2} + y = 0 \qquad \textbf{(C)}\quad 4\,x^2\frac{d^2y}{dx^2} + 8x\,\frac{dy}{dx} - 3\,y = 0$$

Problem 41. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad \frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + 13\,y = 0 \qquad \textbf{(B)}\quad 4\,\frac{d^2y}{dx^2} + 9\,y = 0 \qquad \textbf{(C)}\quad x^2\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + y = 0$$

Problem 42. The ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left(4x^2 + 2\right)\,y = 4x^3\,\cos(2x)$$

has particular solutions $y = \phi_j(x), \quad j = 1, 2, 3$ given by

$$\begin{aligned}
\phi_1(x) &= x^2\,\sin(2x) + 2x\,\sin(2x) + x\,\cos(2x)\\
\phi_2(x) &= x^2\,\sin(2x) - 2x\,\sin(2x) + x\,\cos(2x)\\
\phi_3(x) &= x^2\,\sin(2x) + 2x\,\sin(2x) + 2x\,\cos(2x).
\end{aligned}$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = -\pi^2.$$

Problem 43. Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6\,y = x^3\,\mathrm{e}^{2x}.$$

Problem 44. The ordinary differential equation

$$(x-1)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + y = \frac{(x-1)^2}{x}$$

has complementary solution

$$c_1\,x + c_2\,\mathrm{e}^x.$$

Find a particular solution. What is the general solution?

8 Test 2

Problem 45. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad \frac{d^2y}{dx^2}+\frac{dy}{dx}-2\,y=0 \qquad \textbf{(B)}\quad 4\,\frac{d^2y}{dx^2}+4\,\frac{dy}{dx}-3\,y=0 \qquad \textbf{(C)}\quad 4\,\frac{d^2y}{dx^2}-4\,\frac{dy}{dx}+y=0$$

Problem 46. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad x^2\frac{d^2y}{dx^2}+5x\,\frac{dy}{dx}+13\,y=0 \qquad \textbf{(B)}\quad 4\,x^2\frac{d^2y}{dx^2}+y=0 \qquad \textbf{(C)}\quad 4\,x^2\frac{d^2y}{dx^2}+8x\,\frac{dy}{dx}-3\,y=0$$

Problem 47. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad \frac{d^2y}{dx^2}+4\,\frac{dy}{dx}+13\,y=0 \qquad \textbf{(B)}\quad 4\,\frac{d^2y}{dx^2}+9\,y=0 \qquad \textbf{(C)}\quad x^2\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}+y=0$$

Problem 48. The ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2}-2x\,\frac{dy}{dx}+\left(4x^2+2\right)\,y=4x^3\,\cos(2x)$$

has particular solutions $y=\phi_j(x),\quad j=1,2,3$ given by

$$\begin{aligned}\phi_1(x) &= x^2\,\sin(2x)+2x\,\sin(2x)+x\,\cos(2x)\\ \phi_2(x) &= x^2\,\sin(2x)-2x\,\sin(2x)+x\,\cos(2x)\\ \phi_3(x) &= x^2\,\sin(2x)+2x\,\sin(2x)+2x\,\cos(2x).\end{aligned}$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)^3 \quad\text{and}\quad y'\left(\frac{\pi}{2}\right)=-\pi^2.$$

Problem 49. Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-6\,y=x^3\,e^{2x}.$$

Problem 50. The ordinary differential equation

$$(x-1)\,\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}+y=\frac{(x-1)^2}{x}$$

has complementary solution

$$c_1\,x+c_2\,e^x.$$

Find a particular solution. What is the general solution?

9 Sample Problem Set G

Problem 51. Find the inverse LAPLACE transform

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s^2-1)(s+1)}\right\}.$$

Problem 52. Consider the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4\mathrm{e}^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Find the LAPLACE transform of the solution to this initial value problem.

Problem 53. If

$$\mathcal{L}\{f(t)\} = \frac{s^3}{s^4 - s + 2}$$

compute the LAPLACE transform

$$\mathcal{L}\left\{\mathrm{e}^{-2t} f(t)\right\}.$$

Problem 54. If

$$f(t) = \begin{cases} 3t, & 0 \le t \le 2 \\ 6, & 2 \le t. \end{cases}$$

compute the LAPLACE transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt.$$

Problem 55. Compute and simplify $\mathcal{L}\left\{\mathrm{e}^t \cos t \sin t\right\}$.

Problem 56. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\}.$$

Problem 57. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{(s+1)^2(s-1)}\right\}.$$

Problem 58. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s\,(s^2+4)}\right\}.$$

Problem 59. If

$$y(t) + \int_0^t y(r)\, dr = 1$$

use the LAPLACE transform to find $y(t)$.

10 Sample Problem Set H

Sample Problem Set H contained no problems. It was simply a LAPLACE transform table. It was distributed so people could become familiar with it before the final exam.

11 Final Exam

The original final exam included a table of LAPLACE transforms. The table is not included here.

Problem 60. Find the general solution of each of the ordinary differential equations

$$\textbf{(A)}\quad \frac{dy}{dx}\cos x = y\sin x \qquad \textbf{(B)}\quad x^2\frac{dy}{dx} = y^2 + 3\,xy.$$

Problem 61. The differential equation

$$\left(y - xy^2\right)\,dx + \left(x + x^2y^2\right)\,dy = 0$$

has an integrating factor of the form $x^m y^n$. **(A)** Find the integrating factor. **(B)** Solve the differential equation.

Problem 62. A 50 liter tank initially contains 10 liters of brine of concentration 0.5 gram/liter salt. A brine solution containing 1 gram/liter salt runs into the tank at the rate 4 liter/min. and the well-stired solution is drained off at the rate 2 liter/min. Find the concentration of salt in the brine in the tank at the onset of overflow.

Problem 63. Assume that the acceleration of gravity is 9.8 m/sec^2 so that a 10 kg mass will weigh 98 newtons.

A 10 kg mass is suspended from a spring, stretching it by 0.7 m. The mass is started in motion by pulling it down 0.5 m and releasing it. Assume air resistance has magnitude $90\,\dfrac{dx}{dt}$ newtons where x is the downward displacement of the mass from equilibrium.

(A) Find the equation of motion of the mass and solve it using the appropriate initial values. **(B)** How many times does the mass pass through the equilibrium position after being released?

Problem 64. Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\textbf{(A)}\quad x^2\frac{d^2y}{dx^2} + 11\,x\frac{dy}{dx} + 169\,y = 0 \qquad \textbf{(B)}\quad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5\,y = 0 \qquad \textbf{(C)}\quad x^2\frac{d^2y}{dx^2} - 5\,x\frac{dy}{dx} + 9\,y = 0.$$

Problem 65. Use variation of parameters to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - y = \frac{e^{2x}}{1+e^x}.$$

Problem 66. Use the method of undetermined coefficients to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} + 25\,y = 3\,e^{3x}\,\cos(4x).$$

Problem 67. The ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left(4x^2+2\right)\,y = 4x^3\,\cos(2x)$$

has particular solutions $y = \phi_j(x), \quad j = 1, 2, 3$ given by

$$\begin{aligned}
\phi_1(x) &= x^2\,\sin(2x) + 2x\,\sin(2x) + x\,\cos(2x)\\
\phi_2(x) &= x^2\,\sin(2x) - 2x\,\sin(2x) + x\,\cos(2x)\\
\phi_3(x) &= x^2\,\sin(2x) + 2x\,\sin(2x) + 2x\,\cos(2x).
\end{aligned}$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = -\pi^2.$$

Problem 68. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2\,s^2+3}{s^3+2\,s^2-3\,s}\right\}.$$

Problem 69. Consider the initial value problem

$$\frac{d^2y}{dt^2} + 3\,\frac{dy}{dt} - 4\,y = t\,e^t, \quad y(0) = -2, \;\; y'(0) = 3.$$

Find the Laplace transform of the solution to this initial value problem.

12 Contact Information

The contact information below is accurate as of Jan 11, 2001.

Bent E. Petersen
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605

phone numbers
office (541) 737-5163
home (541) 753-1829
fax (541) 737-0517

bent@alum.mit.edu
petersen@math.orst.edu
http://ucs.orst.edu/~peterseb
http://www.peak.org/~petersen
http://web.orst.edu/~peterseb