Applied Differential Equations – Mth 256

Archive – Winter 1995 Files

Jan 9, 2001

This archive contains two sets of sample problems, two quizzes and the final exam from Mth 256 Winter 1995. The sample problems are from a list I prepared for CRUM in 1994. Most of the CRUM problems were extracted from some of my old tests. The original test instructions, headers and formatting have not been preserved.

Contents

1 Sample Problems - Set 1

Introduction

Informal description of differential equations and initial value problems. Terminology. Explicit and implicit solutions.

No sample problems at this time.

Existence and Uniqueness theorem

Statement of existence and uniqueness theorem. Sketch of idea of proof using Picard iterates or Euler polygonal lines. Examples.

No sample problems at this time.

Separable First Order Equation

Problem 1. Solve (carefully)

$$\frac{dy}{dt} = 2\,t\,y^2, \quad y(1) = 0.$$

Problem 2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x\,y + y}{x}, \quad y(1) = -2.$$

Problem 3. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x\,y + y}{x}\,\log(x), \quad y(1) = -2.$$

Problem 4.

$$\frac{dy}{dt} + 2\,t\,y = 0, \quad y(0) = -2$$

Problem 5.

$$t\,\frac{dy}{dt} + y = 0, \quad y(1) = 4$$

Problem 6.

$$t\,\frac{dy}{dt} - 4\,y = 0, \quad y(2) = -8$$

Problem 7.

$$\frac{dy}{dt} + \tan(t)\,y = 0, \quad y(0) = 2$$

Problem 8.

$$\frac{dy}{dt} + \frac{y}{t} = 0, \quad y(1) = -3$$

Problem 9.

$$\frac{dy}{dt} + \frac{t}{y} = 0, \quad y(1) = -3$$

Problem 10.

$$t^2 \frac{dy}{dt} + y = 0, \quad y(1) = 1$$

Problem 11.

$$\frac{dy}{dt} + \log(t)\, y = 0, \quad y(2) = 1.$$

Problem 12. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} \log(x), \quad y(\mathrm{e}) = \mathrm{e}$$

where e is the base of the natural logarithm (Euler's number).

Problem 13. Find the general solution of the ordinary differential equation

$$xy \frac{dy}{dx} = x^2 + 1$$

Problem 14. Solve the ordinary differential equation

$$y \frac{dy}{dx} = x\, e^{x^2+y^2}.$$

Problem 15. Solve the initial value problem

$$\left(1 + x^2\right) \frac{dy}{dx} = y\, x, \quad y(1) = 2.$$

Problem 16. Solve the initial value problem

$$\frac{dp}{dt} = \mathrm{e}^{2t}\, p(1-p), \qquad p(0) = 0.5.$$

Find

$$\lim_{t \to \infty} p(t).$$

Exact Equations. Integrating Factors

Problem 17. For what value of k is $\left(x^2+y^2\right)^k$ an integrating factor for

$$-y\,dx + x\,dy = 0?$$

Problem 18. For what values of p and q is $x^p y^q$ an integrating factor for the ordinary differential equation

$$(6y^2+3y-4xy)\,dx + (-3x^2+3x+8xy)\,dy = 0?$$

Problem 19. The differential equation

$$\left(y-xy^2\right)\,dx + \left(x+x^2y^2\right)\,dy = 0$$

has an integrating factor of the form $x^m y^n$. **(A)** Find the integrating factor. **(B)** Solve the differential equation.

Problem 20. Find the general solution of the ordinary differential equation

$$(1+\log(xy))\,dx + \left(1+\frac{x}{y}\right)\,dy = 0.$$

Problem 21. Solve the initial value problem

$$\left(3x^2y^2-2xy^3-2x-1\right)dx + \left(2x^3y-3x^2y^2-8y^3-y+1\right)dy = 0, \qquad y(0)=2.$$

Problem 22. Find an integrating factor which depends only on y and then solve the differential equation

$$(2y+y^2-6xy)\,dx \;+\; (4x+3xy-6x^2)\,dy = 0.$$

Problem 23. The ordinary differential equation

$$\left(6\,xy+y^2\right)\,dx + \left(9\,x^2-6+4\,xy\right)\,dy = 0$$

has an integrating factor μ depending only on y. Find the integrating factor and then solve the ordinary differential equation.

Problem 24. For what values of m and n is $x^m y^n$ an integrating factor for the differential equation

$$(3y+12xy^2)dx + (4x+15x^2y)dy = 0.$$

Problem 25. Solve the initial value problem

$$(4x + 4y + 3)dx + (4x - 6y - 2)dy = 0, \qquad y(2) = 1.$$

Problem 26. Solve the *exact* ordinary differential equation

$$(2xy^3 - y^2 - 2)\,dx + (3x^2y^2 - 2xy + 3)\,dy = 0.$$

Problem 27. The ordinary differential equation

$$(6\,y - 4\,y^2)\,dx + (9\,x - 8\,x\,y)\,dy = 0$$

has an integrating factor of the form

$$\mu = x^m\,y^n.$$

Find m and n and then solve the given ordinary differential equation.

Problem 28. The ordinary differential equation

$$(9\,y^2 + 18\,x^2y^2 + 4\,xy^3 + x)\,dx + (18\,xy + 6\,y^2)\,dy = 0$$

has an integrating factor depending only on x. Find this integrating factor and then solve the given ordinary differential equation.

Linear Equations. Integrating Factor

Problem 29.

$$\frac{dy}{dt} = \frac{y}{t-3} + t^2$$

Problem 30.

$$\frac{dy}{dt} = \cot(t)\,y + \sin(t)$$

Problem 31.

$$\frac{dy}{dt} = \frac{y}{t} + \sin\left(t^2\right)$$

Problem 32. Find a function $y(t)$ such that

$$y(t) + \int_0^t y(s)\,ds = t^2.$$

Problem 33. Solve the initial value problem

$$x\,\frac{dy}{dx} + 4\,y = x, \qquad y(1) = -1.$$

Problem 34. Solve the initial value problem

$$\left(1 + x^2\right)\frac{dy}{dx} + \left(3x + \frac{1}{x}\right)y = 6x + 2, \quad y(1) = 2.$$

Problem 35. Solve the initial value problem

$$\frac{dy}{dx} + y\cos(x) = \cos(x), \qquad y(0) = 3.$$

Problem 36. Solve the initial value problem

$$x^2\,\frac{dy}{dx} + 4\,y = 1, \qquad y(1) = 0.$$

Problem 37. Solve the initial value problem

$$x\,\frac{dy}{dx} + 2\,y\log x = 4\,\log x, \qquad y(1) = -1.$$

Problem 38. Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} - (\tan x)y = \sec x.$$

Mixing Problems

Problem 39. A 50 liter tank initially contains 10 liters of brine of concentration 0.5 gram/liter salt. A brine solution containing 1 gram/liter salt runs into the tank at the rate 4 liter/min. and the well-stirred solution is drained off at the rate 2 liter/min. Find the concentration of salt in the brine in the tank at the onset of overflow.

Problem 40. A large tank contains 80 gallons of brine of concentration 1.621 oz/gal salt. Brine of concentration 2.121 oz/gal salt flows into the tank at 3 gal/min. The well-mixed solution is drawn off at the rate of 4 gal/min. When will the brine in the tank reach a *concentration* of 2.013 oz/gal salt?

Problem 41. A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

Problem 42. A tank initially contains 100 L of brine of concentration 0.6 g/L salt. Brine of concentration 2.1 g/L runs into the tank at 6.0 L/min and the well-mixed solution is drained off at 4.0 L/min. Find the concentration of salt in the tank at the moment that the tank contains 220 L brine.

Problem 43. A 200 gal tank initially contains 100 gal fresh water. Brine of concentration 1.2 oz/gal salt flows into the tank at the rate 3 gal/min. The well–mixed solution is drawn off at the rate 2 gal/min. Find the *concentration* of salt in the tank at the very moment that it begins to overflow.

Problem 44. A 200 L tank initially contains 100 L of brine of concentration 3 g/L salt (i.e., 3 grams salt per liter water). Brine of concentration 5 g/L salt runs into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the solution in the tank at the moment that the tank begins to overflow.

Problem 45. A brine solution consisting of 0.06 oz/gal salt dissolved in water flows into a large tank at the rate 3.0 gal/min. The solution inside the tank is kept well-mixed and flows out of the tank at the rate 2.0 gal/min. If the tank initially contains 50.0 gal of brine of concentration 0.03 oz/gal determine the amount of salt in the tank after t minutes. When will the concentration of salt in the tank reach 0.05 oz/gal? Assume the tank is so large that it does not overflow.

Problem 46. A tank of volume 400 L initially contains 200 L of brine of concentration $\frac{1}{5}$ g/L of salt. Brine of concentration 1 g/L flows into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the tank at the moment that it overflows.

Problem 47. A tank initially contains 200 L of brine of concentration $\frac{1}{5}$ g/L of salt. Brine of concentration $\frac{1}{2}$ g/L flows into the tank at 2 L/min. The well–mixed solution is drawn off at the rate 3 L/min. Find the *concentration* of salt in the tank after 100 minutes.

Newton's Law of Cooling

Problem 48. Consider a cup of coffee in a room of temperature $A = 70°$ F. The cup is sitting on a small heating pad which is supposed to keep the coffee warm. If T is the temperature of the coffee then

$$\frac{dT}{dt} = -k\,(T - A) + U$$

where U is a constant depending on the heater and the cup and k is a constant depending on the cup. Initially the temperature of the coffee is 183° F, 5 minutes later the temperature is 155° F, and an additional 5 minutes later the temperature is 135° F.

Find the temperature T as a function of time. Compute

$$\lim_{t\to\infty} T(t).$$

Problem 49. A cup of coffee initially at 190° F is brought into a room at 65° F. After 2 minutes the temperature of the coffee is 145° F. Predict the temperature of the coffee an additional minute later.

If T is the temperature of the coffee then according to Newton's law of cooling

$$\frac{dT}{dt} = -k\,(T - A)$$

where A is the temperature of the room. Note we may assume that A is constant since the room has much higher heat capacity than the coffee.

Problem 50. A thermometer initially reading 62° F is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads 167° F. After an additional 1 second it reads 179° F. If A denotes the temperature of the coffee, T denotes the temperature reading of the thermometer and t denotes time in seconds then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. We regard the temperature A of the coffee also as constant. Find the temperature of the coffee.

Problem 51. A thermometer is brought into a certain room. The room has temperature $A = 25°$ C. If T is the temperature displayed by the thermometer then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads 21.4° C. An additional 20 seconds later it reads 23.4° C. What was the initial reading on the thermometer at the time that it was first brought into the room?

Problem 52. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing sinusoidally, say

$$A = A_0 + A_1 \cos(\omega t)$$

where A_0, A_1 and $\omega > 0$ are constants, and t is time. According to Netwon's law of cooling we have

$$\frac{dT}{dt} = -k\,(T - A)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1, ω and k. (Do not neglect the arbitrary constant.) Which part of your solution represents the steady–state? What is the amplitude, period and phase of the steady–state solution?

Problem 53. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing linearly (for a while at least), say

$$A = A_0 + A_1\,t$$

where A_0 and A_1 are constants, and t is time. According to Netwon's law of cooling we have

$$\frac{dT}{dt} = -k\,(T - A)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1 and k. (Do not neglect the arbitrary constant.)

Problem 54. Consider a cup of coffee in a room of temperature A. If T is the temperature of the coffee then

$$\frac{dT}{dt} = -k\,(T - A)$$

where k is a constant depending on the cup. Initially the temperature of the coffee is 183° F, 3 minutes later the temperature is 155° F, and an additional 3 minutes later the temperature is 135° F.

Find the temperature T as a function of time. Compute the temperature of the room.

Radioactive Decay

Problem 55. A certain radioactive substance decays to 85 % of its original mass in 36 hours. Find the half–life.

Homogeneous Equations

Problem 56.

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

Problem 57.

$$\frac{dy}{dt} = \frac{t^2 + y^2}{t^2}$$

Problem 58.

$$\frac{dy}{dt} = \frac{y - 4\,t}{t - y}$$

Problem 59.

$$\frac{dy}{dt} = \frac{y^2 + 2\,t\,y}{t^2}$$

Problem 60. Find the general solution of each of the ordinary differential equations

$$\textbf{(A)} \quad \frac{dy}{dx}\cos x = y\,\sin x \qquad \textbf{(B)} \quad x^2\,\frac{dy}{dx} = y^2 + 3\,xy.$$

Problem 61. Find the general solution of the ordinary differential equation

$$xy\frac{dy}{dx} = x^2 + y^2.$$

Problem 62. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}, \qquad y(1) = 2.$$

Problem 63. Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}.$$

Problem 64. Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \cot(y/x).$$

Miscellaneous substitutions

Problem 65. Solve the initial value problem

$$\frac{dy}{dx} = \frac{(x + y)^2 - 3}{2}, \quad y(0) = 0.$$

Problem 66. Solve the initial value problem

$$\frac{dy}{dx} = \frac{(3\,x + 2\,y)^2 - 3}{2}, \quad y(0) = 1.$$

Problem 67. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \big(x + y + 2\big)\big(x + y\big).$$

Problem 68. Solve the initial value problem

$$\frac{dy}{dx} = 2 - (2x - y)^2, \qquad y(0) = 1.$$

Hint: The substitution $z = 2x - y$ is illuminating.

Problem 69.

$$\frac{dy}{dt} = (t + y)^2$$

Problem 70. Substitute $w = \mathrm{e}^y$ to solve

$$\mathrm{e}^y \frac{dy}{dt} + \mathrm{e}^y = \mathrm{e}^t.$$

Problem 71. Solve the Bernoulli ordinary differential equation

$$\frac{dy}{dt} - y = t\,y^2.$$

(Hint: Let $w = y^{-1}$.)

Problem 72. Use the substitution $w = 2\,x + 3\,y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(2x + 3y)^2 - 4\,(2x + 3y) + 4}{6\,(2x + 3y)}.$$

Problem 73. Make the substitution $z = x + y$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{\big(x + y - 1\big)\big(x + y\big)}{2x + 2y + 1}$$

Problem 74. Use the substitution $y = x^2 w$ to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{2y^2 + x^3}{xy}.$$

Problem 75. Solve the ordinary differential equation

$$\frac{dy}{dx} = (x+y)^2 + 2\,(x+y).$$

Problem 76. Solve the ordinary differential equation

$$\frac{dy}{dx} = (x+y)^2 + 3\,(x+y) + 1.$$

Orthogonal trajectories

Problem 77. Find the family of orthogonal trajectories to the one-parameter family of hyperbolas given by $2y^2 - x^2 = \alpha$.

Problem 78. Find the orthogonal trajectories for the family of ellipses

$$2y^2 + x^2 = \alpha \quad (\alpha \text{ is an arbitrary parameter}).$$

Problem 79. Find the family of orthogonal trajectories to the one-parameter family of cubics

$$y = \alpha x^3, \qquad \alpha = \text{ arbitrary constant.}$$

Problem 80. Consider the 1-parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1 \qquad\qquad \alpha \text{ a constant (the parameter).}$$

Find the 1-parameter family of orthogonal trajectories.

Problem 81. Given the one–parameter family of curves

$$x^2 + y^2 = \alpha + 2\,\log(x) \qquad (\alpha \text{ is the parameter})$$

find the one–parameter family of orthogonal trajectories.

Problem 82. Given the 1–parameter family of curves

$$y^2 = \alpha\, e^{x^2+y^2}, \quad (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

Problem 83. Given the 1–parameter family of curves

$$x^2 + \alpha\, y^2 = \alpha, \quad (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

2 Sample Problems - Set 2

Linear differential operators, Operational Calculus

No sample problems at this time.

Solutions of Second Order Linear Equations, Examples

Problem 84. For what value of λ is $y = x^{\lambda}\log(x)$ a solution of the ordinary differential equation $x^2y'' - 5xy' + 9y = 0$?

Problem 85. The ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{1/2}, \quad x > 0$$

has a particular solution of the form $Ax^{1/2}$ where A is a constant. **(A)** Find the constant A. **(B)** Does the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^{-1}, \quad x > 0$$

have a solution of the form Ax^{-1}? Explain your answer.

Basic Properties of Homogeneous Linear Equations

No sample problems at this time.

Complex Numbers. Complex Exponential

No sample problems at this time.

Reduction of Order

Problem 86. The ordinary differential equation

$$\left(x^3 - x^2\right)\frac{d^2y}{dx^2} - \left(x^3 + 2\,x^2 - 2\,x\right)\frac{dy}{dx} + 2\left(x^2 + x - 1\right)y = 0$$

has the solution $y_1(x) = x\,\mathrm{e}^x$. Use reduction of order to find a solution y_2 such that $\{y_1,\, y_2\}$ is a fundamental solution set for the given ordinary differential equation.

Problem 87. The ordinary differential equation

$$x\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} - y = 0$$

has the solution $y_1(x) = x\,\mathrm{e}^x$. Use reduction of order to find a solution y_2 such that $\{y_1,\, y_2\}$ is a fundamental solution set for the given ordinary differential equation.

Homogeneous Linear Equations with Constant Coefficients and Homogeneous Cauchy–Euler Equations

Problem 88. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $x^2\,\dfrac{d^2y}{dx^2}+11\,x\,\dfrac{dy}{dx}+169\,y=0$ **(B)** $\dfrac{d^2y}{dx^2}+4\,\dfrac{dy}{dx}+5\,y=0$ **(C)** $x^2\,\dfrac{d^2y}{dx^2}-5\,x\,\dfrac{dy}{dx}+9\,y=0.$

Problem 89. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2\,y=0$ **(B)** $4\,\dfrac{d^2y}{dx^2}+4\,\dfrac{dy}{dx}-3\,y=0$ **(C)** $4\,\dfrac{d^2y}{dx^2}-4\,\dfrac{dy}{dx}+y=0$

Problem 90. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $x^2\dfrac{d^2y}{dx^2}+5x\,\dfrac{dy}{dx}+13\,y=0$ **(B)** $4\,x^2\dfrac{d^2y}{dx^2}+y=0$ **(C)** $4\,x^2\dfrac{d^2y}{dx^2}+8x\,\dfrac{dy}{dx}-3\,y=0$

Problem 91. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $\dfrac{d^2y}{dx^2}+4\,\dfrac{dy}{dx}+13\,y=0$ **(B)** $4\,\dfrac{d^2y}{dx^2}+9\,y=0$ **(C)** $x^2\dfrac{d^2y}{dx^2}-x\,\dfrac{dy}{dx}+y=0$

Problem 92. Find the general solution (*in real form*) for each of the following ordinary differential equations.

(A) $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=0$ **(B)** $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+3y=0$ **(C)** $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+5y=0$

(D) $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+4y=0$ **(E)** $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+3y=0$ **(F)** $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+5y=0$

(G) $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}=0$ **(H)** $\dfrac{d^2y}{dx^2}+4y=0$ **(I)** $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+4y=0$

Problem 93. Find the general solution **(in real form)**:

$$\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} - 4\,y = 0$$

$$\frac{d^2y}{dx^2} + 4\,y = 0$$

$$\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} + 4\,y = 0$$

$$\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} + 13\,y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6\,y = 0$$

Problem 94. Find the general solution **(in real form)**:

$$x^2\,\frac{d^2y}{dx^2} - 8x\,\frac{dy}{dx} = 0$$

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} - 9\,y = 0$$

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + 9\,y = 0$$

$$x^2\,\frac{d^2y}{dx^2} - 5x\,\frac{dy}{dx} + 9\,y = 0$$

$$x^2\,\frac{d^2y}{dx^2} - 5x\,\frac{dy}{dx} + 25\,y = 0$$

$$x^2\,\frac{d^2y}{dx^2} - 6\,y = 0$$

Basic Properties of Nonhomogeneous Linear Equations

Problem 95. The ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left(4x^2 + 2\right)\,y = 4x^3\,\cos(2x)$$

has particular solutions $y = \phi_j(x), \quad j = 1, 2, 3$ given by

$$\begin{aligned}
\phi_1(x) &= x^2 \sin(2x) + 2x \sin(2x) + x \cos(2x) \\
\phi_2(x) &= x^2 \sin(2x) - 2x \sin(2x) + x \cos(2x) \\
\phi_3(x) &= x^2 \sin(2x) + 2x \sin(2x) + 2x \cos(2x).
\end{aligned}$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = -\pi^2.$$

Variation of Parameters, Duhamel's Formula,

and Method of Undetermined Coefficients

Problem 96. Use variation of parameters to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - y = \frac{e^{2x}}{1 + e^x}.$$

Problem 97. Use the method of undetermined coefficients to find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} + 25\,y = 3\,e^{3x}\,\cos(4x).$$

Problem 98. The ordinary differential equation

$$(x - 1)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + y = \frac{(x - 1)^2}{x}$$

has complementary solution

$$c_1\,x + c_2\,e^x.$$

Find a particular solution. What is the general solution?

Problem 99. Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6\,y = x^3\,e^{2x}.$$

Problem 100. Find a particular solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x\left(e^{2x} + e^{-2x}\right).$$

Problem 101. Solve the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Problem 102. Consider the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(x^2 + 2\right)y = x^3 \sec x, \quad x \geq 0.$$

Given that the complementary solution is

$$c_1 x \cos x + c_2 x \sin x$$

use variation of parameters to find the general solution.

Problem 103. Use variation of parameters to find the general solution of the ordinary differential equation

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} - y = 2\,x + 4\,x^{-1}.$$

Problem 104. Use the method of undetermined coefficients (judicious guessing) to find a particular solution:

$$\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} - 10\,y = x^2\,e^{2x}$$

$$\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} + 25\,y = 4x\,e^{3x}\sin(4x)$$

Falling Bodies and similar physical problems

No sample problems at this time.

Mechanical Vibrations, Elementary Electrical Circuits,

Damped Free Vibrations,

Forced Vibrations and Resonance

Problem 105. Assume that the acceleration of gravity is 9.8 m/sec^2 so that a 10 kg mass will weigh 98 newtons.

A 10 kg mass is suspended from a spring, stretching it by 0.7 m. The mass is started in motion by pulling it down 0.5 m and releasing it. Assume air resistance has magnitude $90\,\dfrac{dx}{dt}$ newtons where x is the downward displacement of the mass from equilibrium.

(A) Find the equation of motion of the mass and solve it using the appropriate initial values. **(B)** How many times does the mass pass through the equilibrium position after being released?

Basic Properties of Laplace Transform

Problem 106. Use the table to compute the Laplace transforms

$$\textbf{(A)}\quad \mathcal{L}\left\{t^2 e^{3t}\right\} \qquad \textbf{(B)}\quad \mathcal{L}\left\{e^{2(t-1)}\right\} \qquad \textbf{(C)}\quad \mathcal{L}\left\{(t+1)^3\right\}.$$

Problem 107. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2s^2+7s+3}{(s-1)(s+1)(s+2)}\right\}.$$

Problem 108. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^2-4}{(s+1)(s^2+4)}\right\}.$$

Problem 109. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{e^{-\pi s/2}\,\frac{s-2}{s^2-4s+13}\right\}.$$

Problem 110. Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{2\,s^2+3}{s^3+2\,s^2-3\,s}\right\}.$$

Problem 111. Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s^2-1)(s+1)}\right\}.$$

Problem 112. If

$$\mathcal{L}\{f(t)\} = \frac{s^3}{s^4 - s + 2}$$

compute the LAPLACE transform

$$\mathcal{L}\left\{e^{-2t} f(t)\right\}.$$

Problem 113. If

$$f(t) = \begin{cases} 3t, & 0 \le t \le 2 \\ 6, & 2 \le t. \end{cases}$$

compute the LAPLACE transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Problem 114. Compute and simplify $\mathcal{L}\left\{e^t \cos t \sin t\right\}$.

Problem 115. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\}.$$

Problem 116. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{(s+1)^2(s-1)}\right\}.$$

Problem 117. Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s\,(s^2+4)}\right\}.$$

Problem 118. Compute the inverse Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{5\,s^2+5\,s-6}{s(s+1)(s-2)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{6\,s^2-7\,s+3}{s^2(s-1)}\right\}$$

Laplace Transform and Initial Value Problems

Problem 119. Consider the initial value problem

$$\frac{d^2y}{dt^2} + 3\,\frac{dy}{dt} - 4\,y = t\,\mathrm{e}^t, \quad y(0) = -2, \;\; y'(0) = 3.$$

Find the Laplace transform of the solution to this initial value problem.

Problem 120. Consider the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4\mathrm{e}^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Find the Laplace transform of the solution to this initial value problem.

Problem 121. Find the Laplace transform of the solution to the initial value problem

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 3y = \begin{cases} 2 & \text{if } 0 < t < 1 \\ t & \text{if } 1 < t < 4 \\ -1 & \text{if } 4 < t \end{cases}$$

$$y(0) = 1, \quad y'(0) = -2.$$

(Do not solve the differential equation).

Problem 122. Find the Laplace transform of the solution of the following initial value problem:

$$5\,\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2\,y = \begin{cases} 0 \text{ if } 0 \le t < 2 \\ 2 \text{ if } 2 \le t < 3 \\ 0 \text{ if } 3 \le t \end{cases} \qquad y(0) = -2, \;\; y'(0) = 1.$$

Convolution, Impulses and Generalized Functions

Problem 123. If

$$y(t) + \int_0^t y(r)\,dr = 1$$

use the Laplace transform to find $y(t)$.

Linear Systems of Differential Equations

No sample problems at this time.

Eigenvalues, Eigenvectors and Normal Modes

No sample problems at this time.

3 Test 1

Problem 124. The differential equation

$$x^2 y^3\, dx + x\left(1 + y^2\right)\, dy = 0$$

has an integrating factor of the form $\mu(x, y) = x^p y^q$. Find p and q.

Problem 125. Solve

$$(\sin y - y\, \sin x)\; dx + (x\, \cos y + \cos x)\; dy = 0.$$

Problem 126. Solve the initial value problem

$$\frac{dy}{dx} + y\, \tan(x) = \sec(x), \quad y(\frac{\pi}{4}) = 2\sqrt{2}.$$

$\int \tan(x)\, dx = c + \log\lvert\sec(x)\rvert$	$\int \sec(x)\, dx = c + \log\lvert\sec(x) + \tan(x)\rvert$
$\int \tan^2(x)\, dx = c + \tan(x) - x$	$\int \sec^2(x)\, dx = c + \tan(x)$

Problem 127. A 400 L tank initially is full of brine of concentration 0.8 g/L. Brine of concentration 0.2 g/L flows into the tank at 2 L/min and the well-mixed solution is drawn off at the same rate.

(A) Find the concentration of brine in the tank in g/L as a function of the time in minutes.

(B) How many minutes does it take for the concentration to reach 0.5 g/L?

Problem 128. Find the orthogonal trajectories for the family of ellipses

$$2\,x^2 + y^2 = 2\alpha\, x \qquad (\alpha \text{ is a parameter}).$$

Problem 129. Consider the initial value problem

$$\begin{cases} \dfrac{dp}{dt} = g(t)\, p\,(1 - p) \\ p(t_0) = p_0 \end{cases}$$

where g is a function continuous on $[t_0, \infty)$.

(A) Find the solution $p(t)$. Your solution will contain a *definite* integral of g.

(B) Find

$$\lim_{t\to\infty} p(t)$$

given that

$$\int_{t_0}^{\infty} g(t)\,dt = \log 2.$$

4 Test 2

Problem 130. Find the general solution (in real form)

(A) $$\frac{d^2y}{dx^2} - 5\,\frac{dy}{dx} + 6\,y = 0$$

(B) $$\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} = 0$$

(C) $$\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + 13\,y = 0$$

(D) $$\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + 4\,y = 0$$

(E) $$\frac{d^4y}{dx^4} + 8\,\frac{d^2y}{dx^2} + 16\,y = 0$$

(F) $$\frac{d^4y}{dx^4} - 16\,y = 0$$

Problem 131. Find the general solution (in real form)

(A) $$x^2\,\frac{d^2y}{dx^2} + 2\,x\,\frac{dy}{dx} - 6\,y = 0$$

(B) $$x^2\,\frac{d^2y}{dx^2} + 3\,x\,\frac{dy}{dx} + y = 0$$

(C) $$x^2 \frac{d^2y}{dx^2} + 5\,x\,\frac{dy}{dx} + 5\,y = 0$$

(D) $$x^2 \frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + 4\,y = 0$$

Problem 132. The linear ordinary differential equation

$$2\,x^2 \frac{d^2y}{dx^2} - 4\,x\,\frac{dy}{dx} + \left(2\,x^2 + 4\right)\,y = 4\,x^3\,\cos(x)$$

has the complimentary solution

$$c_1\,x\,\cos(x) + c_2\,x\,\sin(x).$$

Use the method of variation of parameters to find a particular solution of the given inhomogeneous linear ordinary differential equation.

Problem 133. Use the method of undetermined coefficients to find a particular solution of the linear ordinary differential equation

$$\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + 5\,y = (2x+1)\,\mathrm{e}^{-x}\cos(2x).$$

Problem 134. Use the method of undetermined coefficients to find a particular solution of the linear ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2\,y = x\,\mathrm{e}^{x} + x^2\,\mathrm{e}^{2x}.$$

5 Final Exam

The final exam included a Laplace transform table. The table has been omitted in this archive.

Problem 135. Solve the initial value problem

$$\frac{dy}{dx} = \frac{1+y^2}{xy}, \quad y(1) = 2\sqrt{2}.$$

Problem 136. Find the Laplace transforms

(A). $\mathcal{L}\left\{\mathrm{e}^{2t}\cos(3t)\right\}$

(B). $\mathcal{L}\{t\,\sin(t)\}$

Problem 137. Find the inverse Laplace transforms:

(A). $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-1}\right\}$

(B). $\mathcal{L}^{-1}\left\{\dfrac{s^2+3\,s-2}{(s+1)(s^2+1)}\right\}$

Problem 138. Find the Laplace transform of the solution to the initial value problem:

$$2\,\frac{d^2y}{dt^2}-3\,\frac{dy}{dt}+y=\cos(2x),\quad y(0)=1,\quad y'(0)=-1.$$

Problem 139. Here are some trial particular solutions:

(1). $(Ax+B)\mathrm{e}^{-2x}\cos(3x)+(Cx+D)\mathrm{e}^{-2x}\sin(3x)$

(2). $(Ax+B)\mathrm{e}^{3x}\cos(2x)+(Cx+D)\mathrm{e}^{3x}\sin(2x)$

(3). $x(Ax+B)\mathrm{e}^{-2x}\cos(3x)+x(Cx+D)\mathrm{e}^{-2x}\sin(3x)$

(4). $x^2(Ax+B)\mathrm{e}^{-2x}\cos(3x)+x^2(Cx+D)\mathrm{e}^{-2x}\sin(3x)$

(5). $(Ax+B)\mathrm{e}^{-2x}$

(6). $x(Ax+B)\mathrm{e}^{-2x}$

(7). $x^2(Ax+B)\mathrm{e}^{-2x}$

(8). None of the above.

For each of the ordinary differential equations below choose the correct form of a particular solution from the list above. Indicate your choice by circling the appropriate number. *Do not compute the constants* A, B, C nor D.

(A). Choice: 1 2 3 4 5 6 7 8 $\dfrac{d^2y}{dt^2}+4\,\dfrac{dy}{dt}+13\,y=x\,\mathrm{e}^{-2x}\,\sin(3x)$

(B). Choice: 1 2 3 4 5 6 7 8 $\dfrac{d^2y}{dt^2}+4\,\dfrac{dy}{dt}+13\,y=x\,\mathrm{e}^{-2x}$

(C). Choice: 1 2 3 4 5 6 7 8 $\dfrac{d^2y}{dt^2} + 4\,\dfrac{dy}{dt} + 4\,y = x\,\mathrm{e}^{-2x}$

(D). Choice: 1 2 3 4 5 6 7 8 $\dfrac{d^2y}{dt^2} + 4\,\dfrac{dy}{dt} + 3\,y = x\,\mathrm{e}^{-2x}$

Problem 140. The ordinary differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2\,y = x^2+1$$

has the complementary solution

$$c_1\,x + c_2\,\left(x^2-1\right).$$

Use variation of parameters to find a particular solution.

Problem 141. A tank of volume V_0 intially is full of fresh water (no salt). At a certain time brine of concentration 2 oz/gal salt starts flowing into the tank at 1 gal/min and the well–mixed solution is simultaneously drawn off at 2 gal/min. After 6 minutes the concentration of salt in the outflow is found to be 0.2 oz/gal. Find the volume V_0 of the tank.

6 Contact Information

The contact information below is accurate as of Jan 9, 2001.

Bent E. Petersen	phone numbers
Department of Mathematics	office (541) 737-5163
Oregon State University	home (541) 753-1829
Corvallis, OR 97331-4605	fax (541) 737-0517

bent@alum.mit.edu
petersen@math.orst.edu
http://ucs.orst.edu/~peterseb
http://www.peak.org/~petersen
http://web.orst.edu/~peterseb