Suppose a tank, containing a certain fluid, has an outlet near the bottom. If h is the head, that is, the height of the fluid surface above the outlet, then TORRICELLI's principle states that the outflow velocity at the outlet is equal to the free-fall velocity of a particle at height h, that is, $\sqrt{2gh}$. This principle leads to the ODE

$$A(h)\frac{dh}{dt} = -\gamma\, a\sqrt{2gh}$$

where

$A(h)$	=	horizontal cross-sectional area of the tank at height h
a	=	cross-sectional area of the outlet
g	=	acceleration of gravity, 32.1 ft/sec^2
γ	=	BORDA's factor

Note BORDA's factor accounts for contraction of the (assumed smooth) outflow stream. It is about 0.60 for water (77% contraction). BORDA was a contemporary of E. TORRICELLI, 1608–1647.

Problem 0114 − 1. Consider a tank with a constant horizontal cross-sectional area of 28 ft^2 and a height of 12 ft above an outlet. Suppose the outlet has a cross-sectional area of 0.00545 ft^2. If the tank is initially full of water, how long does it take (in seconds) to drain down to the level of the outlet?

Problem 0114 − 2. In problem 1, in addition to everything else assume water runs into the tank at the rate 0.062 ft^3/sec. Find the corresponding differential equation. Find the head corresponding to the equilibrium solution (when inflow balances outflow). (Note: One can show that h approaches the equilibrium value as $t \to \infty$).

Problem 0114 − 3. Repeat problem 1, but this time assume the outlet has a valve which causes its effective cross-section to vary periodically

$$a(t) = \frac{0.00545\left(1+\cos(\frac{t}{200})\right)}{2}.$$

(Note: You have to find a root of a complicated expression in t. You might try a graphical approach. As a hint, the answer is probably well over 14,000 seconds.)

Problem 0114 − 4. Consider two tanks, each full of water. Each tank has constant cross-sectional area. The tanks have the same volume. Each tank has an outlet in the bottom and the two outlets have the same cross-sectional area. Let $T1$ (respectively, $T2$) be the length of time required to drain the first (respectively, the second) tank. Compute the ratio $T1/T2$ if the first tank is four times as tall as the second tank.

Problem 0114 − 5. Consider the tank whose surface consists of the curve $y = cx^4$ rotated about the z-axis (considered vertical), where $0 \le x \le d$. Here c and d are positive constants. Show that

$$A(h) = \pi h^{1/2} c^{-1/2}.$$

Comment on what this fact has to do with designing a clepsydra (water clock).