Consider a rocket of mass m (including fuel) with velocity v. Let F be the sum of all external forces acting on the rocket. Let $p = mv$ be the momentum of the rocket and let Δt be a short period of time.

During the time Δt the rocket expells exhaust gas of mass $-\Delta m$ with exhaust velocity w_e relative to the rocket. Here $\Delta m < 0$ is the change in the total mass of the rocket. The momentum of this bit of exhaust is $(-\Delta m)(v - w_e)$. By Newton's law of action and reaction the momentum of the rocket changes by $\Delta m(v - w_e)$. Thus

$$\begin{aligned} p + \Delta p &= (m + \Delta m)(v + \Delta v) \\ &= mv + \Delta m(v - w_e) + F\Delta t \end{aligned}$$

These equations yields

$$v\Delta m + m\Delta v + (\Delta m)(\Delta v) = \Delta m(v - w_e) + F\Delta t.$$

If we divide by Δt, pass to the limit as $\Delta t \to 0$ and do a bit of algebra we obtain

$$m\frac{dv}{dt} + w_e\frac{dm}{dt} = F \qquad \text{(rocket equation).}$$

Problem 0125 – 1. Assume $F = 0$, m_0 is the initial mass of the rocket and $m = m_0 - \alpha t$ (α is the burn rate). Show that

$$v - v_0 = w_e \log\left(\frac{m_0}{m}\right).$$

If $\frac{3}{4}$ of the rocket is fuel (mass ratio of 4) and we burn all of it show we obtain a velocity of about $1.4w_e$. In general to achieve a high speed, we need a high mass ratio.

Problem 0125 – 2. Consider a short-range missile fired horizontally. If we ignore gravity but take into account atmospheric drag we obtain

$$m\frac{dv}{dt} + w_e\frac{dm}{dt} = -kv$$

where $k > 0$ is the drag coefficient (generally small). As before assume $m = m_0 - \alpha t$ prior to burnout. If the initial velocity v_0 satisfies

$$v_0 \leq \frac{\alpha w_e}{k}$$

show that

$$v(t) \leq \frac{\alpha w_e}{k}$$

for all time $t \geq 0$. Thus to build a fast missile one needs to have a high exhaust velocity and a very low drag coefficient, in addition to having a high mass ratio.

Problem 0125 – 3. If we assume a constant acceleration of gravity g and fire our missile vertically (not too high in view of the first assumption) we obtain the equations

$$m\frac{dv}{dt} + w_e\frac{dm}{dt} = -kv - mg, \qquad m = m_0 - \alpha t$$

valid prior to burnout. Solve the initial value problem with $v(0) = v_0 \geq 0$.