

## Applied Differential Equations Course Description

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Course title: Applied Differential Equations  
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Credit: 4 quarter hours  
Prerequisite: Vector Calculus 1 (Fourth quarter calculus)

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### CATALOG DESCRIPTION

The OSU Catalog description reads as follows:

*First order linear and nonlinear equations, and second order linear equations. Applications to electrical circuits and mechanical oscillators. Introduction to the Laplace transform and higher order equations. Introduction to linear systems of differential equations, eigenvalues and normal modes. Related matrix and linear algebra concepts. Solution methods and applications appropriate for science and engineering.*

This description may be overly ambitious. In practice we cover little more than the material up to and including the Laplace transform and higher order linear equations. For students who have already had some linear algebra it may be possible to do everything indicated above.

The course is offered in two formats: four hours of lectures per week or three hours of lectures plus one GTA led recitation. The recitation format does provide the students with more help but also makes it unlikely that all the material described above will be covered.

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### STUDENT PREPAREDNESS

The students taking Mth 256 at OSU show a remarkable diversity of mathematical preparation and ability. Some are outstanding. Others are in serious trouble from the first day. The following areas are prevalent problem areas:

- English: Many students do not read problems carefully and can't determine "what's wanted". For example, if a problem is phrased as "Given (A) show (B)" a surprising number of students will waste time proving (A). It's important to emphasize to stop to think before starting to compute.
  - Algebra: One frequently sees  $\frac{c}{a+b}$  simplified as  $\frac{c}{a} + \frac{c}{b}$  and other similar errors.
  - Calculus: Most of the first year of calculus is used in differential equations. The following topics are particularly important.
    - (1) Basic mechanics of differentiation and integration.
    - (2) Chain rule
    - (3) Partial fractions, integration by parts, various other techniques of integration
    - (4) Properties of the logarithm function and the exponential function.
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### COURSE DESCRIPTION AND SAMPLE PROBLEMS

The sample problems listed below have not been edited very carefully. There may be some duplications (not too many I hope). Most of the problems have been used on actual Mth 256 tests. On the tests only the statement of the problem

was used - there was no other indication of the category of the problem. Thus the students were confronted by at least one more difficulty than the reader of this report.

Since I wanted to finish this report in a reasonable time I have not provided sample problems for all sections.

### Introduction

Informal description of differential equations and initial value problems. Terminology. Explicit and implicit solutions.  
No sample problems at this time.

### Existence and Uniqueness theorem

Statement of existence and uniqueness theorem. Sketch of idea of proof using Picard iterates or Euler polygonal lines.  
Examples.

No sample problems at this time.

### Separable First Order Equation

**Problem 1.** Solve (carefully)

$$\frac{dy}{dt} = 2t y^2, \quad y(1) = 0.$$

**Problem 2.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy + y}{x}, \quad y(1) = -2.$$

**Problem 3.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy + y}{x} \log(x), \quad y(1) = -2.$$

**Problem 4.**

$$\frac{dy}{dt} + 2t y = 0, \quad y(0) = -2$$

**Problem 5.**

$$t \frac{dy}{dt} + y = 0, \quad y(1) = 4$$

**Problem 6.**

$$t \frac{dy}{dt} - 4y = 0, \quad y(2) = -8$$

**Problem 7.**

$$\frac{dy}{dt} + \tan(t) y = 0, \quad y(0) = 2$$

**Problem 8.**

$$\frac{dy}{dt} + \frac{y}{t} = 0, \quad y(1) = -3$$

**Problem 9.**

$$\frac{dy}{dt} + \frac{t}{y} = 0, \quad y(1) = -3$$

**Problem 10.**

$$t^2 \frac{dy}{dt} + y = 0, \quad y(1) = 1$$

**Problem 11.**

$$\frac{dy}{dt} + \log(t)y = 0, \quad y(2) = 1.$$

**Problem 12.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} \log(x), \quad y(e) = e$$

where  $e$  is the base of the natural logarithm (Euler's number).

**Problem 13.** Find the general solution of the ordinary differential equation

$$xy \frac{dy}{dx} = x^2 + 1$$

**Problem 14.** Solve the ordinary differential equation

$$y \frac{dy}{dx} = x e^{x^2+y^2}.$$

**Problem 15.** Solve the initial value problem

$$(1+x^2) \frac{dy}{dx} = yx, \quad y(1) = 2.$$

**Problem 16.** Solve the initial value problem

$$\frac{dp}{dt} = e^{2t} p(1-p), \quad p(0) = 0.5.$$

Find

$$\lim_{t \rightarrow \infty} p(t).$$

Exact Equations. Integrating Factors

**Problem 17.** For what value of  $k$  is  $(x^2 + y^2)^k$  an integrating factor for

$$-y dx + x dy = 0?$$

**Problem 18.** For what values of  $p$  and  $q$  is  $x^p y^q$  an integrating factor for the ordinary differential equation

$$(6y^2 + 3y - 4xy) dx + (-3x^2 + 3x + 8xy) dy = 0?$$

**Problem 19.** The differential equation

$$(y - xy^2) dx + (x + x^2y^2) dy = 0$$

has an integrating factor of the form  $x^m y^n$ . **(A)** Find the integrating factor. **(B)** Solve the differential equation.

**Problem 20.** Find the general solution of the ordinary differential equation

$$(1 + \log(xy)) dx + \left(1 + \frac{x}{y}\right) dy = 0.$$

**Problem 21.** Solve the initial value problem

$$(3x^2y^2 - 2xy^3 - 2x - 1) dx + (2x^3y - 3x^2y^2 - 8y^3 - y + 1) dy = 0, \quad y(0) = 2.$$

**Problem 22.** Find an integrating factor which depends only on  $y$  and then solve the differential equation

$$(2y + y^2 - 6xy) dx + (4x + 3xy - 6x^2) dy = 0.$$

**Problem 23.** The ordinary differential equation

$$(6xy + y^2) dx + (9x^2 - 6 + 4xy) dy = 0$$

has an integrating factor  $\mu$  depending only on  $y$ . Find the integrating factor and then solve the ordinary differential equation.

**Problem 24.** For what values of  $m$  and  $n$  is  $x^m y^n$  an integrating factor for the differential equation

$$(3y + 12xy^2)dx + (4x + 15x^2y)dy = 0.$$

**Problem 25.** Solve the initial value problem

$$(4x + 4y + 3)dx + (4x - 6y - 2)dy = 0, \quad y(2) = 1.$$

**Problem 26.** Solve the *exact* ordinary differential equation

$$(2xy^3 - y^2 - 2) dx + (3x^2y^2 - 2xy + 3) dy = 0.$$

**Problem 27.** The ordinary differential equation

$$(6y - 4y^2) dx + (9x - 8xy) dy = 0$$

has an integrating factor of the form

$$\mu = x^m y^n.$$

Find  $m$  and  $n$  and then solve the given ordinary differential equation.

**Problem 28.** The ordinary differential equation

$$(9y^2 + 18x^2y^2 + 4xy^3 + x) dx + (18xy + 6y^2) dy = 0$$

has an integrating factor depending only on  $x$ . Find this integrating factor and then solve the given ordinary differential equation.

### Linear Equations. Integrating Factor

**Problem 29.**

$$\frac{dy}{dt} = \frac{y}{t-3} + t^2$$

**Problem 30.**

$$\frac{dy}{dt} = \cot(t)y + \sin(t)$$

**Problem 31.**

$$\frac{dy}{dt} = \frac{y}{t} + \sin(t^2)$$

**Problem 32.** Find a function  $y(t)$  such that

$$y(t) + \int_0^t y(s) ds = t^2.$$

**Problem 33.** Solve the initial value problem

$$x \frac{dy}{dx} + 4y = x, \quad y(1) = -1.$$

**Problem 34.** Solve the initial value problem

$$(1+x^2) \frac{dy}{dx} + \left(3x + \frac{1}{x}\right) y = 6x + 2, \quad y(1) = 2.$$

**Problem 35.** Solve the initial value problem

$$\frac{dy}{dx} + y \cos(x) = \cos(x), \quad y(0) = 3.$$

**Problem 36.** Solve the initial value problem

$$x^2 \frac{dy}{dx} + 4y = 1, \quad y(1) = 0.$$

**Problem 37.** Solve the initial value problem

$$x \frac{dy}{dx} + 2y \log x = 4 \log x, \quad y(1) = -1.$$

**Problem 38.** Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} - (\tan x)y = \sec x.$$

### Mixing Problems

**Problem 39.** A 50 liter tank initially contains 10 liters of brine of concentration 0.5 gram/liter salt. A brine solution containing 1 gram/liter salt runs into the tank at the rate 4 liter/min. and the well-stirred solution is drained off at the rate 2 liter/min. Find the concentration of salt in the brine in the tank at the onset of overflow.

**Problem 40.** A large tank contains 80 gallons of brine of concentration 1.621 oz/gal salt. Brine of concentration 2.121 oz/gal salt flows into the tank at 3 gal/min. The well-mixed solution is drawn off at the rate of 4 gal/min. When will the brine in the tank reach a *concentration* of 2.013 oz/gal salt?

**Problem 41.** A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

**Problem 42.** A tank initially contains 100 L of brine of concentration 0.6 g/L salt. Brine of concentration 2.1 g/L runs into the tank at 6.0 L/min and the well-mixed solution is drained off at 4.0 L/min. Find the concentration of salt in the tank at the moment that the tank contains 220 L brine.

**Problem 43.** A 200 gal tank initially contains 100 gal fresh water. Brine of concentration 1.2 oz/gal salt flows into the tank at the rate 3 gal/min. The well-mixed solution is drawn off at the rate 2 gal/min. Find the *concentration* of salt in the tank at the very moment that it begins to overflow.

**Problem 44.** A 200 L tank initially contains 100 L of brine of concentration 3 g/L salt (i.e., 3 grams salt per liter water). Brine of concentration 5 g/L salt runs into the tank at 8 L/min. The well-mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the solution in the tank at the moment that the tank begins to overflow.

**Problem 45.** A brine solution consisting of 0.06 oz/gal salt dissolved in water flows into a large tank at the rate 3.0 gal/min. The solution inside the tank is kept well-mixed and flows out of the tank at the rate 2.0 gal/min. If the tank initially contains 50.0 gal of brine of concentration 0.03 oz/gal determine the amount of salt in the tank after  $t$  minutes. When will the concentration of salt in the tank reach 0.05 oz/gal? Assume the tank is so large that it does not overflow.

**Problem 46.** A tank of volume 400 L initially contains 200 L of brine of concentration  $\frac{1}{5}$  g/L of salt. Brine of concentration 1 g/L flows into the tank at 8 L/min. The well-mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the tank at the moment that it overflows.

**Problem 47.** A tank initially contains 200 L of brine of concentration  $\frac{1}{5}$  g/L of salt. Brine of concentration  $\frac{1}{2}$  g/L flows into the tank at 2 L/min. The well-mixed solution is drawn off at the rate 3 L/min. Find the *concentration* of salt in the tank after 100 minutes.

### Newton's Law of Cooling

**Problem 48.** Consider a cup of coffee in a room of temperature  $A = 70^\circ \text{F}$ . The cup is sitting on a small heating pad which is supposed to keep the coffee warm. If  $T$  is the temperature of the coffee then

$$\frac{dT}{dt} = -k(T - A) + U$$

where  $U$  is a constant depending on the heater and the cup and  $k$  is a constant depending on the cup. Initially the temperature of the coffee is  $183^\circ \text{F}$ , 5 minutes later the temperature is  $155^\circ \text{F}$ , and an additional 5 minutes later the temperature is  $135^\circ \text{F}$ .

Find the temperature  $T$  as a function of time. Compute

$$\lim_{t \rightarrow \infty} T(t).$$

**Problem 49.** A cup of coffee initially at  $190^\circ \text{F}$  is brought into a room at  $65^\circ \text{F}$ . After 2 minutes the temperature of the coffee is  $145^\circ \text{F}$ . Predict the temperature of the coffee an additional minute later.

If  $T$  is the temperature of the coffee then according to Newton's law of cooling

$$\frac{dT}{dt} = -k(T - A)$$

where  $A$  is the temperature of the room. Note we may assume that  $A$  is constant since the room has much higher heat capacity than the coffee.

**Problem 50.** A thermometer initially reading  $62^\circ \text{F}$  is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads  $167^\circ \text{F}$ . After an additional 1 second it reads  $179^\circ \text{F}$ . If  $A$  denotes the temperature of the coffee,  $T$  denotes the temperature reading of the thermometer and  $t$  denotes time in seconds then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where  $k$  is a constant. We regard the temperature  $A$  of the coffee also as constant. Find the temperature of the coffee.

**Problem 51.** A thermometer is brought into a certain room. The room has temperature  $A = 25^\circ \text{C}$ . If  $T$  is the temperature displayed by the thermometer then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where  $k$  is a constant. After being in the room for 10 seconds the thermometer reads  $21.4^\circ \text{C}$ . An additional 20 seconds later it reads  $23.4^\circ \text{C}$ . What was the initial reading on the thermometer at the time that it was first brought into the room?

**Problem 52.** Consider an insulated box with internal temperature  $T$ . Assume that the ambient (external) temperature  $A$  is changing sinusoidally, say

$$A = A_0 + A_1 \cos(\omega t)$$

where  $A_0$ ,  $A_1$  and  $\omega > 0$  are constants, and  $t$  is time. According to Newton's law of cooling we have

$$\frac{dT}{dt} = -k(T - A)$$

where  $k$  is a constant depending on the insulation of the box. Find the temperature  $T(t)$  in terms of  $t$ ,  $A_0$ ,  $A_1$ ,  $\omega$  and  $k$ . (Do not neglect the arbitrary constant.) Which part of your solution represents the steady-state? What is the amplitude, period and phase of the steady-state solution?

**Problem 53.** Consider an insulated box with internal temperature  $T$ . Assume that the ambient (external) temperature  $A$  is changing linearly (for a while at least), say

$$A = A_0 + A_1 t$$

where  $A_0$  and  $A_1$  are constants, and  $t$  is time. According to Newton's law of cooling we have

$$\frac{dT}{dt} = -k(T - A)$$

where  $k$  is a constant depending on the insulation of the box. Find the temperature  $T(t)$  in terms of  $t$ ,  $A_0$ ,  $A_1$  and  $k$ . (Do not neglect the arbitrary constant.)

**Problem 54.** Consider a cup of coffee in a room of temperature  $A$ . If  $T$  is the temperature of the coffee then

$$\frac{dT}{dt} = -k(T - A)$$

where  $k$  is a constant depending on the cup. Initially the temperature of the coffee is  $183^\circ$  F, 3 minutes later the temperature is  $155^\circ$  F, and an additional 3 minutes later the temperature is  $135^\circ$  F.

Find the temperature  $T$  as a function of time. Compute the temperature of the room.

### Radioactive Decay

**Problem 55.** A certain radioactive substance decays to 85 % of its original mass in 36 hours. Find the half-life.

### Homogeneous Equations

**Problem 56.**

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

**Problem 57.**

$$\frac{dy}{dt} = \frac{t^2 + y^2}{t^2}$$

**Problem 58.**

$$\frac{dy}{dt} = \frac{y-4t}{t-y}$$

**Problem 59.**

$$\frac{dy}{dt} = \frac{y^2 + 2ty}{t^2}$$

**Problem 60.** Find the general solution of each of the ordinary differential equations

$$\text{(A)} \quad \frac{dy}{dx} \cos x = y \sin x \quad \text{(B)} \quad x^2 \frac{dy}{dx} = y^2 + 3xy.$$

**Problem 61.** Find the general solution of the ordinary differential equation

$$xy \frac{dy}{dx} = x^2 + y^2.$$

**Problem 62.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}, \quad y(1) = 2.$$

**Problem 63.** Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}.$$

**Problem 64.** Find the general solution of the ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \cot(y/x).$$

### Miscellaneous substitutions

**Problem 65.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{(x+y)^2 - 3}{2}, \quad y(0) = 0.$$

**Problem 66.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{(3x+2y)^2 - 3}{2}, \quad y(0) = 1.$$

**Problem 67.** Make the substitution  $z = x + y$  to solve the ordinary differential equation

$$\frac{dy}{dx} = (x + y + 2)(x + y).$$

**Problem 68.** Solve the initial value problem

$$\frac{dy}{dx} = 2 - (2x - y)^2, \quad y(0) = 1.$$

Hint: The substitution  $z = 2x - y$  is illuminating.

**Problem 69.**

$$\frac{dy}{dt} = (t + y)^2$$

**Problem 70.** Substitute  $w = e^y$  to solve

$$e^y \frac{dy}{dt} + e^y = e^t.$$

**Problem 71.** Solve the Bernoulli ordinary differential equation

$$\frac{dy}{dt} - y = t y^2.$$

(Hint: Let  $w = y^{-1}$ .)

**Problem 72.** Use the substitution  $w = 2x + 3y$  to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(2x + 3y)^2 - 4(2x + 3y) + 4}{6(2x + 3y)}.$$

**Problem 73.** Make the substitution  $z = x + y$  to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(x + y - 1)(x + y)}{2x + 2y + 1}$$

**Problem 74.** Use the substitution  $y = x^2 w$  to solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{2y^2 + x^3}{xy}.$$

**Problem 75.** Solve the ordinary differential equation

$$\frac{dy}{dx} = (x + y)^2 + 2(x + y).$$

**Problem 76.** Solve the ordinary differential equation

$$\frac{dy}{dx} = (x + y)^2 + 3(x + y) + 1.$$

### Orthogonal trajectories

**Problem 77.** Find the family of orthogonal trajectories to the one-parameter family of hyperbolas given by  $2y^2 - x^2 = \alpha$ .

**Problem 78.** Find the orthogonal trajectories for the family of ellipses

$$2y^2 + x^2 = \alpha \quad (\alpha \text{ is an arbitrary parameter}).$$

**Problem 79.** Find the family of orthogonal trajectories to the one-parameter family of cubics

$$y = \alpha x^3, \quad \alpha = \text{arbitrary constant.}$$

**Problem 80.** Consider the 1-parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1 \quad \alpha \text{ a constant (the parameter).}$$

Find the 1-parameter family of orthogonal trajectories.

**Problem 81.** Given the one-parameter family of curves

$$x^2 + y^2 = \alpha + 2 \log(x) \quad (\alpha \text{ is the parameter})$$

find the one-parameter family of orthogonal trajectories.

**Problem 82.** Given the 1-parameter family of curves

$$y^2 = \alpha e^{x^2+y^2}, \quad (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

**Problem 83.** Given the 1-parameter family of curves

$$x^2 + \alpha y^2 = \alpha, \quad (\alpha \text{ is the parameter})$$

find the family of orthogonal trajectories.

### Linear differential operators, Operational Calculus

No sample problems at this time.

### Solutions of Second Order Linear Equations, Examples

**Problem 84.** For what value of  $\lambda$  is  $y = x^\lambda \log(x)$  a solution of the ordinary differential equation  $x^2 y'' - 5xy' + 9y = 0$ ?

**Problem 85.** The ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^{1/2}, \quad x > 0$$

has a particular solution of the form  $Ax^{1/2}$  where  $A$  is a constant. **(A)** Find the constant  $A$ . **(B)** Does the ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^{-1}, \quad x > 0$$

have a solution of the form  $Ax^{-1}$ ? Explain your answer.

### Basic Properties of Homogeneous Linear Equations

No sample problems at this time.

### Complex Numbers. Complex Exponential

No sample problems at this time.

### Reduction of Order

**Problem 86.** The ordinary differential equation

$$(x^3 - x^2) \frac{d^2 y}{dx^2} - (x^3 + 2x^2 - 2x) \frac{dy}{dx} + 2(x^2 + x - 1)y = 0$$

has the solution  $y_1(x) = xe^x$ . Use reduction of order to find a solution  $y_2$  such that  $\{y_1, y_2\}$  is a fundamental solution set for the given ordinary differential equation.

**Problem 87.** The ordinary differential equation

$$x \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - y = 0$$

has the solution  $y_1(x) = xe^x$ . Use reduction of order to find a solution  $y_2$  such that  $\{y_1, y_2\}$  is a fundamental solution set for the given ordinary differential equation.

### Homogeneous Linear Equations with Constant Coefficients

### and Homogeneous Cauchy–Euler Equations

**Problem 88.** Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$(A) \quad x^2 \frac{d^2 y}{dx^2} + 11x \frac{dy}{dx} + 169y = 0 \quad (B) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0 \quad (C) \quad x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} + 9y = 0.$$

**Problem 89.** Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$(A) \quad \frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 0 \quad (B) \quad 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 3y = 0 \quad (C) \quad 4 \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0$$

**Problem 90.** Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$(A) \quad x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0 \quad (B) \quad 4x^2 \frac{d^2 y}{dx^2} + y = 0 \quad (C) \quad 4x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} - 3y = 0$$

**Problem 91.** Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$(A) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 13y = 0 \quad (B) \quad 4 \frac{d^2 y}{dx^2} + 9y = 0 \quad (C) \quad x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$$

**Problem 92.** Find the general solution (*in real form*) for each of the following ordinary differential equations.

$$\begin{array}{lll} (A) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0 & (B) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = 0 & (C) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0 \\ (D) \quad x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 4y = 0 & (E) \quad x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 3y = 0 & (F) \quad x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 5y = 0 \\ (G) \quad \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 0 & (H) \quad \frac{d^2 y}{dx^2} + 4y = 0 & (I) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4y = 0 \end{array}$$

**Problem 93.** Find the general solution (**in real form**):

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} = 0$$

$$\frac{d^2 y}{dx^2} - 4y = 0$$

$$\frac{d^2 y}{dx^2} + 4y = 0$$

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$$

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 13y = 0$$

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

**Problem 94.** Find the general solution (**in real form**):

$$x^2 \frac{d^2 y}{dx^2} - 8x \frac{dy}{dx} = 0$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - 9y = 0$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 9y = 0$$

$$x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} + 9y = 0$$

$$x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} + 25y = 0$$

$$x^2 \frac{d^2 y}{dx^2} - 6y = 0$$

### Basic Properties of Nonhomogeneous Linear Equations

**Problem 95.** The ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + (4x^2 + 2)y = 4x^3 \cos(2x)$$

has particular solutions  $y = \phi_j(x)$ ,  $j = 1, 2, 3$  given by

$$\phi_1(x) = x^2 \sin(2x) + 2x \sin(2x) + x \cos(2x)$$

$$\phi_2(x) = x^2 \sin(2x) - 2x \sin(2x) + x \cos(2x)$$

$$\phi_3(x) = x^2 \sin(2x) + 2x \sin(2x) + 2x \cos(2x).$$

Use this data to find the solution of the above ordinary differential equation with initial conditions

$$y\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^3 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) = -\pi^2.$$

### Variation of Parameters, Duhamel's Formula, and Method of Undetermined Coefficients

**Problem 96.** Use variation of parameters to find the general solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} - y = \frac{e^{2x}}{1 + e^x}.$$

**Problem 97.** Use the method of undetermined coefficients to find the general solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 25y = 3e^{3x} \cos(4x).$$

**Problem 98.** The ordinary differential equation

$$(x-1) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \frac{(x-1)^2}{x}$$

has complementary solution

$$c_1 x + c_2 e^x.$$

Find a particular solution. What is the general solution?

**Problem 99.** Find the general solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = x^3 e^{2x}.$$

**Problem 100.** Find a particular solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = x(e^{2x} + e^{-2x}).$$

**Problem 101.** Solve the initial value problem

$$\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + 5y = 4e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

**Problem 102.** Consider the ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + (x^2 + 2)y = x^3 \sec x, \quad x \geq 0.$$

Given that the complementary solution is

$$c_1 x \cos x + c_2 x \sin x$$

use variation of parameters to find the general solution.

**Problem 103.** Use variation of parameters to find the general solution of the ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 2x + 4x^{-1}.$$

**Problem 104.** Use the method of undetermined coefficients (judicious guessing) to find a particular solution:

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 10y = x^2 e^{2x}$$

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 25y = 4x e^{3x} \sin(4x)$$

Falling Bodies and similar physical problems

No sample problems at this time.

Mechanical Vibrations, Elementary Electrical Circuits,

Damped Free Vibrations,

Forced Vibrations and Resonance

**Problem 105.** Assume that the acceleration of gravity is  $9.8 \text{ m/sec}^2$  so that a  $10 \text{ kg}$  mass will weigh  $98 \text{ newtons}$ .

A  $10 \text{ kg}$  mass is suspended from a spring, stretching it by  $0.7 \text{ m}$ . The mass is started in motion by pulling it down  $0.5 \text{ m}$  and releasing it. Assume air resistance has magnitude  $90 \frac{dx}{dt}$  newtons where  $x$  is the downward displacement of the mass from equilibrium.

(A) Find the equation of motion of the mass and solve it using the appropriate initial values. (B) How many times does the mass pass through the equilibrium position after being released?

Basic Properties of Laplace Transform

**Problem 106.** Use the table to compute the LAPLACE transforms

$$\text{(A)} \quad \mathcal{L}\{t^2 e^{3t}\} \quad \text{(B)} \quad \mathcal{L}\{e^{2(t-1)}\} \quad \text{(C)} \quad \mathcal{L}\{(t+1)^3\}.$$

**Problem 107.** Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1} \left\{ \frac{2s^2 + 7s + 3}{(s-1)(s+1)(s+2)} \right\}.$$

**Problem 108.** Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1} \left\{ \frac{s^2 - 4}{(s+1)(s^2 + 4)} \right\}.$$

**Problem 109.** Compute the inverse LAPLACE transform

$$\mathcal{L}^{-1} \left\{ e^{-\pi s/2} \frac{s-2}{s^2 - 4s + 13} \right\}.$$

**Problem 110.** Compute the inverse Laplace transform

$$\mathcal{L}^{-1} \left\{ \frac{2s^2 + 3}{s^3 + 2s^2 - 3s} \right\}.$$

**Problem 111.** Find the inverse LAPLACE transform

$$\mathcal{L}^{-1} \left\{ \frac{s^2 + 1}{(s^2 - 1)(s + 1)} \right\}.$$

**Problem 112.** If

$$\mathcal{L}\{f(t)\} = \frac{s^3}{s^4 - s + 2}$$

compute the LAPLACE transform

$$\mathcal{L}\{e^{-2t}f(t)\}.$$

**Problem 113.** If

$$f(t) = \begin{cases} 3t, & 0 \leq t \leq 2 \\ 6, & 2 \leq t. \end{cases}$$

compute the LAPLACE transform

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

**Problem 114.** Compute and simplify  $\mathcal{L}\{e^t \cos t \sin t\}$ .

**Problem 115.** Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\}.$$

**Problem 116.** Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{s^2 + s + 6}{(s+1)^2(s-1)}\right\}.$$

**Problem 117.** Compute and simplify

$$\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s(s^2+4)}\right\}.$$

**Problem 118.** Compute the inverse Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{5s^2 + 5s - 6}{s(s+1)(s-2)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{6s^2 - 7s + 3}{s^2(s-1)}\right\}$$

### Laplace Transform and Initial Value Problems

**Problem 119.** Consider the initial value problem

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = te^t, \quad y(0) = -2, \quad y'(0) = 3.$$

Find the Laplace transform of the solution to this initial value problem.

**Problem 120.** Consider the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 4e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Find the LAPLACE transform of the solution to this initial value problem.

**Problem 121.** Find the LAPLACE transform of the solution to the initial value problem

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 3y = \begin{cases} 2 & \text{if } 0 < t < 1 \\ t & \text{if } 1 < t < 4 \\ -1 & \text{if } 4 < t \end{cases}$$

$$y(0) = 1, \quad y'(0) = -2.$$

(Do not solve the differential equation).

**Problem 122.** Find the Laplace transform of the solution of the following initial value problem:

$$5 \frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ 2 & \text{if } 2 \leq t < 3 \\ 0 & \text{if } 3 \leq t \end{cases} \quad y(0) = -2, \quad y'(0) = 1.$$

Convolution, Impulses and Generalized Functions

**Problem 123.** If

$$y(t) + \int_0^t y(r) dr = 1$$

use the LAPLACE transform to find  $y(t)$ .

Linear Systems of Differential Equations

No sample problems at this time.

Eigenvalues, Eigenvectors and Normal Modes

No sample problems at this time.

## LAPLACE TRANSFORM TABLE

I usually provide the students with a table of Laplace transforms on any test involving Laplace transforms. A number of students have trouble interpreting the table. More practice might be helpful . . . .

Short Table of LAPLACE Transforms

If  $\mathcal{L}\{f(t)\}(s) = F(s)$  then

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(r) dr \quad (\text{if } \frac{f(t)}{t} \text{ integrable at } 0)$$

$$\mathcal{L}\left\{\int_0^t f(r) dr\right\}(s) = \frac{F(s)}{s}$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\}(s) = sF(s) - f(0) \quad (\text{if } f \text{ cont. on } [0, \infty))$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\}(s) = s^2F(s) - sf(0) - f'(0) \quad (\text{if } f, f' \text{ cont. on } [0, \infty))$$

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s) \quad (u = \text{unit step})$$

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

If  $\mathcal{L}\{f(t)\}(s) = F(s)$  and  $\mathcal{L}\{g(t)\}(s) = G(s)$  then  $\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)$ .

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\cos \omega t\}(s) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\{\sin \omega t\}(s) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\{e^{at} \cos \omega t\}(s) = \frac{s-a}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\{e^{at} \sin \omega t\}(s) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\{\sqrt{t}\}(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}.$$