

Instructions: \implies
If you do not read the instructions, then how will you know what to do? Read them now.

Be sure to write your name in the space above.

- You may use one note-sheet prepared in advance. You must put your name on your note-sheet. Your note-sheet may not be larger than 8.5 by 11 inches (22 by 28 cm). You may write on both sides of your note-sheet.
- You may not use any books nor additional note sheets.
- You may use a calculator, but it is difficult to imagine what you will use it for. Calculators and other equipment may not be shared.
- Write your solutions neatly. Use the backs of the examination sheets for scratch work. There is a blank (mostly) page at the end of the test which may also be used for rough work.
- Partial credit will be assigned only for clear, correct, legible work. Remember the emphasis is on clear correct mathematical arguments. If you tell me that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, you are right, but you will not get much of a grade unless you also tell me convincingly how you deduced it converges.

Problem 1. (40 points).

Part (A): If $a \geq 0$ and $b \geq 0$ compute $(\sqrt{b} - \sqrt{a})^2$ to obtain an upper bound for $2\sqrt{ab}$.

Part (B): If $a_n \geq 0$, $b_n \geq 0$ for each $n \in \mathbb{N}$ and the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge give a careful proof that $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges.

Problem 2. (40 points).

Part (A): State the Heine–Borel theorem.

Part (B): Let $(x_n)_{n \geq 1}$ be a sequence of real numbers and suppose this sequence converges with limit α . Let

$$S = \{\alpha\} \cup \{x_n \mid n \in \mathbb{N}\}.$$

Prove that S is compact by establishing the Heine–Borel property directly.

Problem 3. (40 points). For any real number a define

$$a^+ = \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases}, \quad a^- = \begin{cases} 0 & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

Part (A): Show that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

Let a_n be a real number for each $n \in \mathbb{N}$.

Part (B): Show that the series $\sum_{n=1}^{\infty} a_n$ is *absolutely* convergent if and only if the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are *both* convergent.

Part (C): Show if the series $\sum_{n=1}^{\infty} a_n$ is *conditionally* convergent then the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are *both* divergent.

Problem 4. (150 points). Discuss convergence or divergence of each of the following series. In the case of convergence, indicate if it is absolute or conditional. Give precise (but brief) reasoning in each case.

Part (A):
$$\sum_{n=1}^{\infty} (-1)^n 3^n \frac{(n!)^2}{(2n)!}$$

Part (B):
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Part (C):
$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2 + n} - n \right)^{-n}$$

Part (D):
$$\sum_{n=1}^{\infty} \left(\sqrt{n^2 + n} - n \right)^n$$

Part (E):
$$\sum_{n=1}^{\infty} 2^{-n+\log n}$$

This page is (mostly) blank. Use it for overflow (neatly labelled) and scratch work, but do not remove it.

Key:	Spreadsheet line:
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Please do not write in the boxes to the right. They are for your grades.

1	2	3	4	5	6	7	8	9	10	Total
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Note: There are 4 problems for a total of 270 points.