Mth 311 Review Problems November 17, 1997 [1]

We will have an in–class test Friday November 21. Here is a brief list of *some* of the topics we will have discussed before the test:

1. Set theory and cardinality
2. Sequences and completeness of $\mathbb{R}$
   - Cauchy sequences
   - Bounded monotone sequences
   - Bolzano–Weierstrass theorem
   - $\liminf$ and $\limsup$
3. Series
   - Cauchy criterion
   - Monotone convergence
   - Comparison test
   - Absolute and conditional convergence
   - Limit comparison test
   - Cauchy condensation test
   - Zero test
   - Zero test for positive series
   - Alternating series test (or Leibniz's test)
   - Root test
   - Ratio test
   - Geometric series
   - Abel p-series (or hyperharmonic series)
4. Summation by parts
   - Abel's convergence test

Here are a few sample test problems. Please do not expect a problem for each topic above. After all I need to leave some time to make up the test too!

Some of the problems below are too difficult for a test, but trying to do them will still be good review.

Problem 1. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. Prove if

$$\sum_{n=1}^{\infty} |\, a_{n+1} - a_n \,|$$

converges then the sequence $(a_n)_{n\geq 1}$ converges. (You can of course prove a "stronger" result, but this one has the advantage of being useful.) Sequences which satisfy the hypothesis of this problem are sometimes said to be *absolutely convergent*. This old terminology is not much used any more though.

Problem 2. If

$$a_n = \frac{(3n)!}{(2n)!n!}$$

for each $n \geq 1$ compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Problem 3. Find the binary expansion of $3/7$.

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Problem 4. Let $(s_n)_{n\geq 1}$ be a sequence with $0 \leq s_n \leq s_{n+1}$ for each n. Define

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k, \quad n \geq 1.$$

Prove $(\sigma_n)_{n\geq 1}$ is increasing and $\sigma_n \leq s_n$ for each n.

Problem 5. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers with $a_n \neq 0$ for each $n \geq 1$. Prove

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf_{n\to\infty} \mid a_n \mid^{1/n} \leq \limsup_{n\to\infty} \mid a_n \mid^{1/n} \leq \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Problem 6. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers with $a_n \neq 0$ for each $n \geq 1$. Suppose

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Prove that $\lim_{n\to\infty} \mid a_n \mid^{1/n}$ exists and

$$L = \lim_{n\to\infty} \mid a_n \mid^{1/n}.$$

Problem 7. Compute

$$\lim_{n\to\infty} \left(\frac{(3n)!}{(2n)!n!} \right)^{1/n}.$$

Problem 8. Compute

$$\lim_{n\to\infty} \frac{1}{n} \left(\frac{n!}{\sqrt{n}} \right)^{1/n}.$$

Do not use Stirling's formula.

Problem 9. Let $(s_n)_{n\geq 1}$ be a sequence of real numbers with $s_n \geq 0$ for each n. Let

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k, \quad n \geq 1.$$

If $n < m$ show that

$$\sigma_m \leq \frac{s_1 + \cdots + s_n}{m} + \frac{m-n}{m} \max\{s_{n+1}, \cdots, s_m\}$$

and

$$\min\{s_{n+1}, \cdots, s_m\} \leq \frac{m\sigma_m - n\sigma_n}{m-n}.$$

Conclude

$$\liminf_{n\to\infty}\, s_n \leq \liminf_{n\to\infty}\, \sigma_n \leq \limsup_{n\to\infty}\, \sigma_n \leq \limsup_{n\to\infty}\, s_n.$$

Problem 10. If the limit exists then compute it. Otherwise explain why the limit does not exist –

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

Problem 11. If the limit exists then compute it. Otherwise explain why the limit does not exist –

$$\lim_{n\to\infty} n \sin(1/n).$$

Problem 12. If the limit exists then compute it. Otherwise explain why the limit does not exist –

$$\lim_{n\to\infty} n^3 \sin(1/n) - n^2.$$

In the last three problems you have to "cheat" a bit since you need to make use of properties of $\sin x$ which we have not yet established.

Consider now a few problems concerning convergence of series. In each problem, you should settle the matter by making a few estimates (or sometimes a precise calculation) and then appealing to one of the standard tests. If you can use more than one test, by all means do so just for the practice!

Problem 13. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n^4}{3^n}.$$

Problem 14. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n}.$$

Problem 15. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}.$$

Problem 16. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}.$$

Problem 17. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{n^{2n+1}}.$$

Problem 18. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

Problem 19. Investigate the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^5}.$$

Problem 20. Investigate the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}.$$

Problem 21. Investigate the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n^2-1)}}.$$

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