

## A Model Argument. Euler's Number. <sup>1</sup>

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One of the biggest problems for students in advanced calculus is the requirement to provide proofs. More precisely the problem is figuring out what constitutes a proof. Here I will provide a model argument for sequential convergence. I have given much more detail than is customary in mathematical exposition. You are expected to do likewise. Once you have more experience you can suppress some details, but not yet.

The argument below may seem long and involved, even frightening. On close inspection though you will note that it is actually very simple to understand piece-meal, and after a bit, globally. The only difficult part is producing the argument in the first place. The best way to learn this skill is to read a lot of mathematics and, in particular, to practice writing a great deal.

You will note the argument below is counter-intuitive. I begin by establishing some facts I will use later. This approach is efficient but you may wonder how it is achieved. The short answer is "by rewriting." One first works out a muddled meandering argument. Then one re-organizes it while paying close attention to the flow of logic – "cut-and-paste" is very helpful here. You can "improve" the argument below a great deal in this direction if you wish. Try it.

In mathematical exposition it is useful to pay attention to the English since careful writing can help sharpen and clarify the presentation of ideas. For homework and class assignments it is not necessary to write beautiful English however – for example, you may abuse "thus", "hence", "whence", "wherefor", "thence", "it follows that", "implies", "therefore", "it is seen that", and so forth as much as you wish. Just be sure that what you are describing is really an implication in the mathematical sense – that is, beware of using the language too colloquially. Try to be uncompromisingly mathematical – you can always tastefully ease up a bit once you are able to produce correct arguments.

Here then is the problem we will consider:

*Problem:* Let  $a_n = \sum_{k=0}^n \frac{1}{k!}$  and let  $b_n = \left(1 + \frac{1}{n}\right)^n$ ,  $n \geq 1$ . Show that  $a = \lim_{n \rightarrow \infty} a_n$  and  $b = \lim_{n \rightarrow \infty} b_n$  exist and are equal.

The common value of  $a$  and  $b$  is of course Euler's number  $e$ .

We begin by noting

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<sup>1</sup>This document was created in Scientific Notebook Version 3.0 (for fun) and lightly edited by hand for ordinary  $\text{\LaTeX}2_{\epsilon}$ .

$$((1)) \quad \sum_{k=0}^m 2^{-k} \leq 2.$$

Actually if  $r \neq 1$  then

$$\sum_{k=0}^m r^k = \frac{1 - r^{m+1}}{1 - r}.$$

Thus the sum in (1) is actually  $2 - 2^{-m}$  which is certainly bounded by 2. Next we note

$$((2)) \quad k! \geq 2^{k-1} \text{ for each } k \geq 1.$$

This is certainly true for  $k = 1$ . Let  $k \geq 2$  and assume we have proved (2) for  $k - 1$ . Then  $k! = k(k - 1)! \geq k2^{k-2} \geq 2(2^{k-2}) = 2^{k-1}$ . Now (2) follows by induction.

Consider now the sequence  $(a_n)_{n \geq 1}$  defined by

$$a_n = \sum_{k=0}^n \frac{1}{k!}.$$

Since  $a_{n+1} = a_n + \frac{1}{(n+1)!}$  this sequence is monotone increasing. If  $n \geq 1$  then

$$a_n = 1 + \sum_{k=1}^n \frac{1}{k!} \leq 1 + \sum_{k=0}^{n-1} 2^{-k} \leq 3$$

and so the sequence  $(a_n)_{n \geq 1}$  is bounded above. By completeness of the real numbers it now follows that

$$a = \lim_{n \rightarrow \infty} a_n$$

exists.

Consider next the sequence  $(b_n)_{n \geq 1}$ . From the binomial theorem we have

$$b_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{p_{n,k}}{k!}$$

where

$$p_{n,k} = \frac{n!}{(n-k)!n^k} = \prod_{h=1}^{k-1} \left(1 - \frac{h}{n}\right)$$

if  $1 < k \leq n$  and where  $p_{n,1} = p_{n,0} = 1$ . Clearly we have

$$((3)) \quad 0 < p_{n,k} \leq 1 \quad \text{if } 0 \leq k \leq n.$$

It follows that

$$0 < b_n \leq \sum_{k=0}^n \frac{1}{k!} = a_n \leq 3$$

so the sequence  $(b_n)_{n \geq 1}$  is bounded above. Next observe if  $2 \leq k \leq n$  then

$$p_{n,k} = \prod_{h=1}^{k-1} \left(1 - \frac{h}{n}\right) \leq \prod_{h=1}^{k-1} \left(1 - \frac{h}{n+1}\right) = p_{n+1,k}.$$

It follows that

$$b_{n+1} = \sum_{k=0}^{n+1} \frac{p_{n+1,k}}{k!} \geq \sum_{k=0}^n \frac{p_{n+1,k}}{k!} \geq \sum_{k=0}^n \frac{p_{n,k}}{k!} = b_n$$

and so the sequence  $(b_n)_{n \geq 1}$  is monotone increasing. Once again, by completeness of the real numbers the limit

$$b = \lim_{n \rightarrow \infty} b_n$$

exists. Moreover, since  $b_n \leq a_n$  for each  $n \geq 1$  we conclude that

$$b \leq a.$$

To prove  $a = b$  it remains to show that  $a \leq b$ . For this it suffices to show if  $\epsilon > 0$  then

$$((4)) \quad a_n \leq b + \epsilon.$$

Indeed inequality (4) implies that  $b + \epsilon$  is an upper bound for the sequence  $(a_n)_{n \geq 1}$  which implies  $a \leq b + \epsilon$ . This is always true, but here it is particularly obvious since  $a$  is the least upper bound of the monotone increasing sequence  $(a_n)_{n \geq 1}$ . Once we have for any  $\epsilon > 0$  that  $a \leq b + \epsilon$  then it follows that  $a \leq b$  since the contrary conclusion would lead to a contradiction.

Now we prove inequality (4). Let  $n \geq 1$ . Let  $\epsilon > 0$ . Since  $p_{m,k} = \prod_{h=1}^{k-1} (1 - \frac{h}{m})$  we clearly have  $\lim_{m \rightarrow \infty} p_{m,k} = 1$ . Thus for each  $k$  we have an integer  $N_k$  such that

$$m > N_k \text{ implies } 1 - p_{m,k} < \frac{\epsilon}{3}.$$

Now let  $m > n$ . Then we have

$$b - a_n \geq b_m - a_n = \sum_{k=0}^m \frac{p_{m,k}}{k!} - \sum_{k=0}^n \frac{1}{k!} \geq \sum_{k=0}^n \frac{p_{m,k}}{k!} - \sum_{k=0}^n \frac{1}{k!} = \sum_{k=0}^n \frac{p_{m,k} - 1}{k!}.$$

It follows that

$$a_n \leq b + \sum_{k=0}^n \frac{1 - p_{m,k}}{k!}$$

for any  $m > n$ . In particular if we choose any  $m > \max(n, N_0, \dots, N_n)$  we have

$$a_n \leq b + \frac{\epsilon}{3} \sum_{k=0}^n \frac{1}{k!} \leq b + \epsilon$$

which is what we wanted to prove.

## EPILOGUE

There is an important subtle point in the last part of the argument. The expression for  $b_m$  in the estimate involves  $m+1$  terms. But we are able to discard a number of terms so that only the first (fixed number)  $n+1$  terms need to be estimated. This is important because while we know  $p_{m,k}$  has limit 1 as  $m \rightarrow \infty$ , we have no reason to think the convergence uniform in  $k$ . That is, in general, we may have  $N_k$  grow with  $k$ . In the present case we do not have a problem since we only need to compare  $n+1$  of the  $p_{m,k}$  with 1.