Mth 311 November 7, 1997 [*] | Mth 311 The Zero Convergence Test

The simplest series convergence test is the *zero Test.*

Proposition 1 (Zero Test 1). *If* $\sum_{k=1}^{\infty} a_k$ *converges then* $\lim_{k\to\infty} a_k = 0$.

The converse is true for alternating series (Zero Test 2), though false in general. In the case of series with positive terms we can strengthen the conclusion to obtain a useful test (Zero Test 3).

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ be the n^{th} partial sum. Then

$$a_n = s_n - s_{n-1} \quad \text{for } n \geq 2.$$

Now use the additive property of limits. ☐

The proof also follows directly from the Cauchy criterion. In either case we see we are dealing with a very simple result and we do not expect it to be particularly useful in its present form.

In the case of alternating series we have the converse is true. This fact is quite useful.

Proposition 2 (Zero Test 2). *If* $0 < a_{k+1} \leq a_k$ *for each* k *then*

$$\sum_{k=1}^{\infty} (-1)\, a_k$$

converges if and only if $\lim_{k\to\infty} a_k = 0$.

Proof. Assume $\lim_{k\to\infty} a_k = 0$. We begin by grouping the terms in the Cauchy tails. If $n > m$ then

$$(-1)^{m+1} \sum_{k=m+1}^{n} (-1)^k\, a_k = \begin{cases} (a_{m+1} - a_{m+2}) + \cdots + (a_{n-2} - a_{n-1}) + a_n & \text{if } n-m \text{ odd} \\ (a_{m+1} - a_{m+2}) + \cdots + (a_{n-1} - a_n) & \text{if } n-m \text{ even.} \end{cases}$$

The terms in the parentheses are nonnegative and so the sums on the right are bounded below by $a_{m+1} - a_{m+2}$.

Consider now a different grouping,

$$(-1)^{m+1} \sum_{k=m+1}^{n} (-1)^k\, a_k = \begin{cases} a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{n-1} - a_n) & \text{if } n-m \text{ odd} \\ a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{n-2} - a_{n-1}) - a_n & \text{if } n-m \text{ even.} \end{cases}$$

On the right, in either case, we are subtracting nonnegative terms. Thus a_{m+1} is an upper bound for the sums. We conclude

$$0 \leq a_{m+1} - a_{m+2} \leq (-1)^{m+1} \sum_{k=m+1}^{n} (-1)^k\, a_k \leq a_{m+1}.$$

It follows that

$$\left| \sum_{k=m+1}^{n} (-1)^k\, a_k \right| \leq a_{m+1}.$$

Now the Cauchy criterion implies convergence of the series. ☐

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Example 1 (Harmonic series). Let $\lfloor x \rfloor$ denote the greatest integer in x. Let $n > 1$ and let $m = \lfloor \log_2 n \rfloor$ and let s_n be the n^{th} partial sum of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Then

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{k} &\geq \sum_{k=1}^{2^m} \frac{1}{k} \\ &= 1 + \sum_{j=1}^{m} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} \\ &\geq 1 + \sum_{j=1}^{m} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j} \\ &= \frac{2+m}{2}. \end{aligned}$$

Thus we have the estimate

$$\sum_{k=1}^{n} \frac{1}{k} \geq \frac{2 + \lfloor \log_2 n \rfloor}{2}$$

which shows the divergence of the harmonic series.

Actually one can show

$$\sum_{k=1}^{n} \frac{1}{k} = \epsilon_n + \gamma + \log n$$

where γ is the Euler–Mascheroni constant and $\log n$ is the natural logarithm of n.

The divergence of the harmonic series shows we have no hope of a general converse to *Zero Test 1*.

The counting argument used above to prove divergence of the harmonic series can be used to obtain a stronger version of *Zero Test 1* in the case of certain series of positive terms.

Proposition 3 (Zero Test 3). *If* $0 < a_{k+1} \leq a_k$ *for each* k *and* $\sum_{k=1}^{\infty} a_k$ *converges then* $\lim_{k\to\infty} k a_k = 0$.

Note that *Zero Test 3* is strong enough to imply the divergence of the harmonic series.

Proof. Let $\epsilon > 0$. By the Cauchy criterion there is m such that $n > m$ implies

$$(n - m)a_n \leq \sum_{k=m+1}^{n} a_k < \frac{\epsilon}{2}.$$

Now by *Zero Test 1* we may choose $N \geq m$ such that $n > N$ implies

$$a_n < \frac{\epsilon}{2m}.$$

Then

$$n > N \quad \text{implies} \quad n a_n \leq m a_n + \frac{\epsilon}{2} < \epsilon.$$

□

Example 2. Consider the log-harmonic series

$$\sum_{k=2}^{n} \frac{1}{k \log k}.$$

Let $m = \lfloor \log_2 n \rfloor$. Then for the n^{th} partial sum we have

$$\begin{aligned} s_n = \sum_{k=2}^{n} \frac{1}{k \log k} &\geq \sum_{k=2}^{2^m} \frac{1}{k \log k} \\ &= \sum_{j=1}^{m} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k \log k} \\ &\geq \sum_{j=1}^{m} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j \log 2^j} \\ &= \frac{1}{2 \log 2} \sum_{j=1}^{m} \frac{1}{j}. \end{aligned}$$

It follows by our estimate for the harmonic series that

$$s_n \geq \frac{2 + \lfloor \log_2 \lfloor \log_2 n \rfloor \rfloor}{2}$$

and so s_n diverges to $+\infty$ roughly as $\log \log n$.

Thus the log-harmonic series provides an example for which *Zero Test 3* fails.

Remark 3. Let $s = \sum_{k=1}^{\infty} (-1)^k a_k$ be a convergent alternating series (as in *Zero Test 2*) with partial sums $s = \sum_{k=1}^{n} (-1)^k a_k$. In the proof of *Zero Test 2* we obtained the estimate

$$0 \leq (-1)^{m+1} (s_n - s_m) \leq a_{m+1}$$

for $n > m$. If we pass to the limit we obtain

$$0 \leq (-1)^{m+1} (s - s_m) \leq a_{m+1}.$$

Thus the error in approximating s by s_m is bounded by a_{m+1} and the signed error $s - s_m$ alternates in sign. This last fact means that s_m is alternately too large and too small as an estimate of s. Thus we expect the average

$$t_m = \frac{s_m + s_{m+1}}{2} = s_m + \frac{1}{2} (-1)^{m+1} a_{m+1}$$

would be a better estimator of s.

Exercise 1. Do a few numerical tests, for example for the alternating harmonic series, to compare the estimators s_m and t_m.

Exercise 2. These notes were thrown together very quickly and may contain some errors or nonoptimal choices. Read them over carefully and improve what you can. Investigate the possibility of a Zero Test 4 involving $(n \log n) a_n$.

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