Elementary Numerical Analysis – Mth 351

Archive – Summer 1999 Files

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This archive contains the sample problems and tests from Mth 256 Summer 1999. The original test instructions, headers and formatting have not been preserved.

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1 Midterm Test

Problem 1. Consider the cubic polynomial

$$p(x) = 3\,x^3 - 17\,x^2 + 26\,x - 9.$$

If we compute $p(x)$ at several integer points we obtain a table

$x =$	-2	-1	0	1	2	3	4	5	6
$p(x) =$	-153	-55	-9	3	-1	-3	15	71	183

Part (A): Based on the table above which intervals of length 1 can you guarantee contain a root of $p(x)$? Why?

Part (B): Use the initial guess $x_0 = 2$ and compute the first two Newton iterates x_1 and x_2.

Part (C): Given $x_3 = 1.83561601595\cdots$ estimate the error in x_2. Why do you expect your estimate to be a good one?

Problem 2. Let

$$f(x) = e^{3x^5 \tan(x) - 3x^6 - x^8}.$$

The Taylor polynomial of $f(x)$ about the origin of degree ≤ 15 is

$$1 + \frac{2}{5}x^{10} + \frac{17}{105}x^{12} + \frac{62}{945}x^{14}.$$

Let $p(x)$ be the Taylor polynomial of $f(x)$ about the origin of degree ≤ 11.

Part (A): Find $p(x)$.

Part (B): Given $\left| \frac{1}{12!} f^{(12)}(\xi) \right| \leq \frac{3}{5}$ for $0 \leq \xi \leq 1$ estimate the (absolute) error in $p(x)$ as an approximation for $f(x)$, for each x with $0 \leq x \leq 1$, by using the Taylor remainder.

Part (C): Compute $B = \int_0^1 p(x)\, dx$. If we use B as an estimate for $\int_0^1 f(x)\, dx$ estimate the (absolute) error in B by using the results above.

Problem 3. It is not difficult to see

$$(2727273)(12674893) = 34567893456789 \quad \text{and} \quad (3718196)(9296953) = 34567893456788$$

and therefore

$$(2727273)(12674893) - (3718196)(9296953) = 1.$$

Part (A): What answer are you likely to get if you compute this number on a 10 or a 12 digit calculator?

Part (B): How many digits of precision (significant digits) do you think

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Problem 4. Consider the 1 step back, 2 steps forward, method for estimating the third derivative $f^{(3)}(a)$ given by

$$D_{h,3}f(a) = \frac{-f(a-h) + 3f(a) - 3f(a+h) + f(a+2h)}{h^3}.$$

Find the order of this method, that is, find p such that the error in $D_{h,3}f(a)$ as an approximation of $f^{(3)}(a)$ is $\mathcal{O}(h^p)$.

Problem 5. Let f be a function such that $f(1) = 2$, $f(3) = 4$, $f(2) = 5$ and $f(-1) = 8$.

Part (A): Find the Newton divided differences $f[1]$, $f[1,3]$, $f[1,3,2]$ and $f[1,3,2,-1]$.

Part (B): Use the result of part (A) to find the Lagrange interpolation polynomial $p(x)$ for $f(x)$ with nodes at $1,3,2,-1$.

Part (C): If $\left| f^{(4)}(\xi) \right| \le \dfrac{6}{25}$ for ξ in the interval $[-1,3]$ estimate the error in $p(x)$ as an approximation to $f(x)$.

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2 Final Exam

Problem 6. Find A, B, C and D such that

$$s(x) = \begin{cases} 24 + Ax + Bx^2 + 2x^3, & 1 \le x < 2 \\ 48 - 32x + Cx^2 + Dx^3, & 2 \le x \le 4 \end{cases}$$

Problem 7. For a certain function f we have the Newton divided differences

$$f[1] = -36, \qquad f[1,2] = 36, \qquad f[1,2,4] = -18$$

$$f[1,2,4,7] = 4, \quad f[1,2,4,7,8] = 1.$$

Part (A): Find $f[4,2,7,1]$.

Part (B): Find the interpolation polynomial $p(x)$ of degree ≤ 4 for $f(x)$ with nodes at $1,2,4,7$ and 8.

Part (C): If we use $p(x)$ to estimate $f(x)$ what is our estimate of $f(3)$.

Problem 8. Let $T_n(f)$ denote the trapezoidal quadrature formula for the function f, with n equal length subintervals. In class we saw if n is even and we perform a certain Richardson extrapolation we obtain Simpson's rule in the form

$$S_n(f) = \frac{4}{3}T_n(f) - \frac{1}{3}T_{n/2}(f).$$

If n is divisible by 4 we can repeat the process and we obtain Boole's rule in the form

$$B_n(f) = \frac{16}{15}S_n(f) - \frac{1}{15}S_{n/2}(f).$$

Suppose $\int_0^1 f(x)\, dx = 0.49979926915939992 78258\cdots$ and suppose $T_1(f) = 0.85826285$, $T_2(f) = 0.57944883$, and $T_4(f) = 0.51924378$.

Part (A): Compute $S_4(f)$ and $B_4(f)$.

Problem 9. Consider the quadrature method

$$I(f) = \frac{5}{18} f\left(\frac{1}{2}\,\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}}\right) + \frac{4}{9} f\left(\frac{1}{2}\right) + \frac{5}{18} f\left(\frac{1}{2}\,\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}}\right)$$

for approximating the integral $\int_0^1 f(x)\ \mathrm{d}x$.

Part (A): Given the table

$\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}} = \frac{1}{2} \pm \frac{1}{10}\sqrt{15}$	$\left(\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}\right)^4 = \frac{31}{100} \pm \frac{2}{25}\sqrt{15}$
$\left(\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}\right)^2 = \frac{2}{5} \pm \frac{1}{10}\sqrt{15}$	$\left(\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}\right)^5 = \frac{11}{40} \pm \frac{71}{1000}\sqrt{15}$
$\left(\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}\right)^3 = \frac{7}{20} \pm \frac{9}{10}\sqrt{15}$	$\left(\frac{1}{2}\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}\right)^6 = \frac{61}{250} \pm \frac{63}{1000}\sqrt{15}$

compute the exact error in $I\left(x^p\right)$ for $p = 0, 1, 2, 3, 4, 5, 6$. (Simplify your calculations by noting some cancellations.)

Part (B): If $P(x)$ is a polynomial of degree $\leq m$ what is the largest value of m for which you can guarantee that $I(P) = \int_0^1 P(x)\ \mathrm{d}x$, that is, for which $I(P)$ is exact?

Problem 10. Write a brief essay describing a topic in Mth 351 which you enjoyed the most, or found the most interesting. Include some technical details and use reasonably good English.

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3 Assignment 1 - Taylor polynomials

Let

$$f(x) = \mathrm{e}^{1-\cos(x)}.$$

The TAYLOR polynomial at 0 of degree 11 for f may be shown to be

$$p(x) = 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{720}x^6 - \frac{43}{40320}x^8 - \frac{127}{1814400}x^{10}.$$

We are interested in knowing how well $p(x)$ approximates $f(x)$ for $0 \leq x \leq 1.100$. To estimate TAYLOR's remainder we need the 12^{th} derivative of f. If we compute this derivative and estimate it we obtain

$$-59{,}000 \leq f^{(12)}(x) \leq 8{,}850 \quad \text{for } 0 \leq x \leq 1.10.$$

(You may use this fact.)

Problem 11. Find a constant C such that

$$|\, f(x) - p(x)\, | \leq C x^{12} \quad \text{for } 0 \leq x \leq 1.10$$

Hence estimate the maximum absolute error on the interval [0, 1.10], that is,

$$\max_{0 \leq x \leq 1.10} |\, f(x) - p(x)\, | \,.$$

Problem 12. By plotting $f(x) - p(x)$ (you can use a spreadsheet, for example), or by some other method, estimate the maximum of $|\, f(x) - p(x)\, |$ for $0 \leq x \leq 1.10$. How well does this actual error compare with the error estimate in problem (1)? Discuss the distribution of the absolute error $|\, f(x) - p(x)\, |$ for $0 \leq x \leq 1.10$.

Problem 13. Argue that $0 \leq f(x) \leq 8$ for all x.

Problem 14. Compute $p(30)$ and use the result to argue, without additional calculation, that $|\, f(x) - p(x)\, |$ must be as large as 4.0×10^{10} for some x with $0 \leq x \leq 30$. What do you think about using $p(x)$ to approximate $f(x)$ on the interval [0, 30]?

Instructions: *The work you turn in must be correct and be neatly and logically presented. Tables of numerical results and graphs should be labelled. Each solution should include comments written in decent English. You may write about anything at all, but you may find it prudent to include comments about actual errors compared to estimated errors, distribution of errors, or any surprises or difficulties you encountered while doing the problem. Be creative! If you write a program include your code. If you use an interactive maple (or whatever) session include a listing of the session. Throw in anything else that's relevant, but be sure it's neat, labelled and bears your name.*

These instructions will not be repeated on future assignments. They apply however to all our assignments.

4 Assignment 2 - Discrete derivatives

```
> restart; Digits:=16:
```

Here's the well-known central difference estimate for the the *first* derivative of the function f at a.

```
> E1:=(f,a,h)->(f(a+h)-f(a-h))/(2*h);
```

$$E1 := (f,\, a,\, h) \rightarrow \frac{1}{2}\, \frac{f(a+h) - f(a-h)}{h}$$

The derivative of $\cos(x)$ at $\pi/2$ is -1. This gives us a check on `E1`.

```
> E1(cos,Pi/2,.001);
```

$$-.9999998333333415$$

Now here is a slightly less well-known central difference estimate for the *fourth* derivative of f at a:

```
> E2:=(f,a,h)->(f(a+2*h)-4*f(a+h)+6*f(a)-4*f(a-h)+f(a-2*h))/(h^4);
```

$$E2 := (f, a, h) \to \frac{f(a+2\,h) - 4\,f(a+h) + 6\,f(a) - 4\,f(a-h) + f(a-2\,h)}{h^4}$$

The fourth derivative of $\cos(x)$ at π is -1. This gives a check on `E2`.

```
> E2(cos,Pi,.01);
```

$$-.9999833800000000$$

Note Maple gives only 8 significant figures. Thus we lost 8 significant figures in the calculation. Let's try an even smaller step size.

```
> E2(cos,Pi,.001);
```

$$-.9997000000000000$$

Note the round-off in computing the numerator of `E2` is at worst $C u$ where C is a constant depending on f and on a and u is the *unit round.* If p is the order of the truncation error in `E1` then we see the total is at worst (roughly)

```
> C*u/(h^4) + M*h^p;
```

$$\frac{C\,u}{h^4} + M\,h^p$$

For small h the first term dominates and is very large. Thus it is not hard to see this function of h has a minimum for a unique $h > 0$. If we know C and M roughly we can use the methods of calculus to determine an optimum step size h! Most of the time though the best we can say is do not use too small a step size.

Problem 15. Plot the absolute error $|1 + \mathtt{E2}(\cos, \pi, h)|$ for small $h > 0$ (Perhaps $0 < h < .05$).

Problem 16. Select a few step sizes in the range where your graph looks reasonably like $M\,h^p$. Use the values of the absolute errors at these points to estimate the order of `E2`.

Problem 17. Use Taylor polynomials to verify (hopefully) your guess.

Remark: Your results will vary depending on the precision you use. Most spreadsheets will probably use about 16 decimal digits precision. If you use Maple you can request a specific precision by setting `Digits`. However Maple will do its best to maintain as much precision as possible, so some intermediate results may be computed and used with greater accuracy than you intended.

5 Assignment 3 - Interpolation

Let

$$f(x) = (x+1)^9, \quad |\,x\,| \le 1.$$

Part (A): Suppose we use the (curious) choice of nodes

$$-1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

and let $P_A(x)$ be the interpolation polynomial for $f(x)$ of degree ≤ 6 with these nodes. A calculation shows

$$\begin{aligned} p_A(x) &= 140.4374990\,x^6 + 163.4882813\,x^5 + 67.26562600\,x^4 + 85.71777318\,x^3 \\ &+ 47.29687480\,x^2 + 6.793945370\,x + 1.000000000 \end{aligned}$$

Plot $f(x) - p_A(x)$ to get an idea of how the interpolation error behaves.

Part (B): Suppose we use the Chebyshev nodes

$$\cos\left(\frac{(2k+1)}{14}\pi\right), \quad k = 0, \cdots, 6,$$

that is,

$$.9749279122, .7818314824, .4338837394, 0, -.4338837394, -.7818314824, -.9749279122$$

and let $P_B(x)$ be the interpolation polynomial for $f(x)$ of degree ≤ 6 with these nodes. A calculation shows

$$\begin{aligned} p_B(x) &= 99.75000006\,x^6 + 191.1875000\,x^5 + 118.1250018\,x^4 + 51.0781237\,x^3 \\ &+ 36.9843744\,x^2 + 13.1289055\,x + 1.0000005 \end{aligned}$$

Plot $f(x) - p_B(x)$ to get an idea of how the interpolation error behaves.

Part (C): Suppose we use the equidistributed nodes

$$-1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$$

and let $P_C(x)$ be the interpolation polynomial for $f(x)$ of degree ≤ 6 with these nodes. Compute $P_C(x)$ and plot $f(x) - p_C(x)$ to get an idea of how the interpolation error behaves.

Part (D): Discuss the size and distribution of the errors observed above. Do your observations agree with your expectations?

6 Assignment 4 - Quadrature

If we ask Maple to evaluate an integral, Maple always attempts to perform a symbolic integration. Thus we have

```
> int(t^3/(exp(t)-1),t);
```

$$-\frac{1}{4}t^4 + t^3\ln(1-e^t) + 3\,t^2\,\mathrm{polylog}(2,\,e^t) - 6\,t\,\mathrm{polylog}(3,\,e^t) + 6\,\mathrm{polylog}(4,\,e^t)$$

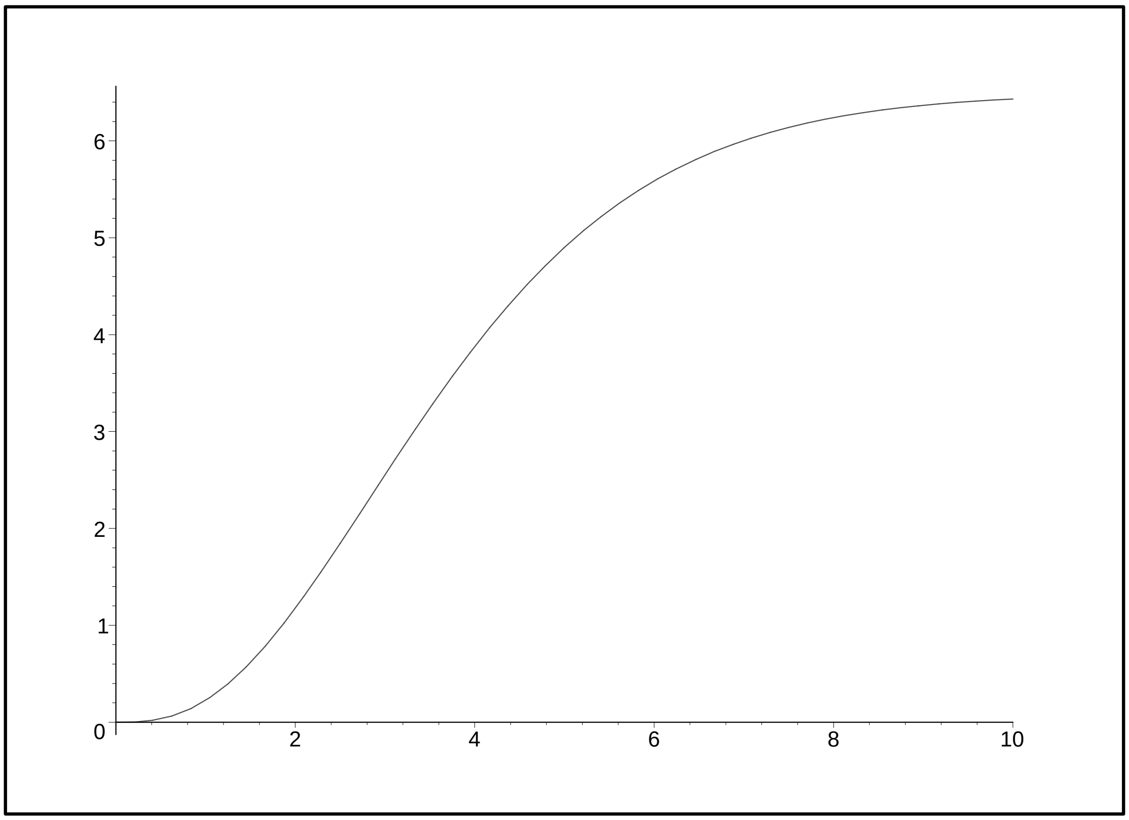
This looks nice but the polylog function is not all that well known. One difficulty with the polylog function is that the definition involves analytic continuation, and therefore complex numbers may arise unexpectedly in expressions for the definite integral corresponding to the indefinite integral shown above. If we just want an approximate numerical value for the definite integral we can use the unevaluated integral Int() and feed the result directly to evalf(). This tells Maple to avoid symbolic integration and instead to use numerical quadrature. The definite integral

```
> f:=x->evalf(Int(t^3/(exp(t)-1),t=0..x));
```

$$f := x \to \mathrm{evalf}\left(\int_0^x \frac{t^3}{e^t - 1}\, dt\right)$$

occurs in the Debye model for the heat capacity of a body and we certainly would like to be able to compute it. Here is a plot of the integral

```
> plot(f,0..10);
```



In this assignment you will test the (compound) trapezoidal and Simpson's rules to see how well they do in approximating A = f(4.0)

```
> Digits:=20:  A:=f(4.0); Digits:=10:
```

$$A := 3.8770541615311946229$$

Problem 1: Estimate A by using the (compound) trapezoidal method with 2, 4 and 8 intervals (so 3, 5 and 9 points, or step size 2, 1 and 1/2). In each case use the value of A given above to compute the error. Comment on how the error varies with the step-size. Does it agree with your expectations.

Problem 2: Estimate A by using the (compound) Simpson's method with 2, 4 and 8 intervals (so 3, 5 and 9 points, or step size 2, 1 and 1/2). In each case use the value of A given above to compute the error. Comment on how the error varies with the step-size. Does it agree with your expectations.

Comments: The integrand above is defined by the limit at t=0. This limit is 0. In Maple you can define the integrand as follows

```
> g:=proc(t)
> if t=0 then
> 0;
> else
> t^3/(exp(t)-1);
> fi;
> end:
```

Then

```
> f:=x->evalf(Int(g(t),t=0..x)); f(4);
```

$$f := x \to \text{evalf}\left(\int_0^x \text{g}(t)\,dt\right)$$

$$3.877054162$$

and g is the function you use in the trapezoidal and Simpson's formulae. The functions trapezoid() and simpson() are provided in Maple's student package (do a with(student); command) but they do not work well with integrals that have an apparent singularity as here. You will have to define your own procedure, or write out the formulae explicitly. Alternately you can do the calculations on a calculator.

Note Maple's int() function is not troubled by the apparent singularity at 0

```
> int(t^3/(exp(t)-1),t=0..4); evalf(%);
```

$$-64 + 64\ln(e^4 - 1) + 64\,I\,\pi + 48\,\text{dilog}(1 - e^4) - 24\,\text{polylog}(3,\, e^4) + 6\,\text{polylog}(4,\, e^4) - \frac{1}{15}\pi^4$$

$$3.877054134 + .3\,10^{-6}\,I$$

though we do get a spurious imaginary part. All the imaginary numbers should cancel here. What we see comes from roundoff error. The only reason we have to be careful about the definition of g above is because g(0) enters explicitly into the trapezoidal and Simpson's rules.

You can also use piecewise() sometimes, but in the present case it does not work well.

7 Contact Information

The contact information below is accurate as of Feb 14, 2001.

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