Mth 351 Sample Problems | Name: ID:

Bent Petersen `351u2002_samples_1.tex` July 22, 2002 Time: xx minutes.

Instructions: $\Longrightarrow$
If you do not read the instructions, then how will you know what to do? Read them now.

Be sure to write your name in the space above.

- *You may use one* *note-sheet* *prepared in advance. You must put your name on your note-sheet, but* *do not* *turn in your note-sheet. Your note-sheet must be letter size,* 8.5×11 *inches, or A4 paper,* 21×29.7 *cm, or smaller. You may write on both sides of your note-sheet.*
- *Note-sheets may not be shared. If you do not bring a note-sheet you will have to do without any help notes.*
- *You may not use any books, notebooks nor additional note-sheets.*
- *You may use a calculator. Calculators and other equipment may not be shared.*
- *For work-out problems sketch your work neatly. Highlight your answer by drawing a frame around it. Scratch out irrelevant or incorrect work so it will be clear what you are submitting as a solution. Give exact answers when possible. Simplify your answer when reasonable to do so. Partial credit will be assigned only for relevant, clear, correct, legible work. If you do not show some relevant work or explain your solution, your grade may be 0.*
- *For multiple-choice problems indicate your choice in the answer box provided. You need not show any work nor offer any explanations for your answer. If you need to do some work, you may do it in the space provided, if any, or on the back of the examination sheets, but your work will not be graded.* ***You will be graded only on the letter you select and put in the provided answer box.*** *Note this test does not use a scantron.*
- *Use the backs of the examination sheets for scratch work.*

Please note $\log(x)$ *means the natural logarithm of* x *on this test.*

Problem 1. *(25 points if correct, 0 points if wrong).* Use the bisection method to estimate the root of the polynomial $p(x) = 3\,x^4 - 14\,x^3 + 24\,x^2 - 4\,x - 3$ in the interval $[0, 1]$ with an error of at most $\frac{1}{8}$.

A.) $\frac{1}{2}$ **B.)** $\frac{5}{8}$
C.) $\frac{3}{4}$ **D.)** $\frac{7}{8}$ **E.)** None of the above.

☐ $\leftarrow$*Letter corresponding to your answer to problem 1.*

Problem 2. Let $f(x) = \exp(x) - x^2$. Prove that f has precisely one real root. Explain why the root must be in the interval $(-1, 0)$.

Problem 3. Let $p(x) = 2x^3 - 3x^2 + 6x + 5$. The polynomial $p(x)$ has a root in the interval $[-1, 0]$. **(A)** Suppose we bisect the interval. Which half of the interval can we guarantee contains a root. Why? **(B)** Bisect the subinterval known to contain a root. Which sub-sub-interval can we now guarantee contains the root? **(C)** If we use the midpoint of this last interval as an estimate of the root find an bound for the error that we make.

Problem 4. Show analytically that the polynomial

$$p(x) = x^3 - 6x^2 + 9x - 5$$

has a root in the interval $[4, 5]$. Suppose we bisect the interval. (A) Which half of the interval can we guarantee contains a root? (B) If we use the midpoint of the subinterval containing the root to estimate the root give an upper bound for the error.

Problem 5. Find two interval of length 1 with integer endpoints, each interval guaranteed to contain a root of $f(x) = \exp(x) - 6x$. Be sure to justify your answer (without finding the roots). Use an initial guess $x_0 = 0$ and apply Newton's iteration once to obtain a new estimate x_1 for one of the roots.

Problem 6. Let $p(x) = x^3 + x^2 + 3x - 4$. Let $x_0 = 1$ be an initial guess to a root. **(A)** Use Newton's method (twice) to compute successive approximations $x_1,\ x_2$ to a root. **(B)** Estimate the error in the root estimate x_1. **(C)** Given that $p(x)$ has only one real root, $.8663697595\ldots$, find the actual errors in x_1 and x_2. How do they compare with your estimate of the error in x_1?

Problem 7. Let $a > 0$ and let $f(x) = x^3 - a$. Suppose we decide to estimate the cube root $a^{1/3}$ by applying Newton's method to estimate the positive root of $f(x)$. For $n \geq 0$ find an expression for the $(n+1)^{st}$ iterate x_{n+1} in terms of x_n. Simplify.

Problem 8. One of the roots of the polynomial $p(x) = x^2 - x - 1$ is the golden ratio $(1 + \sqrt{5})/2 = 1.61803398\cdots$. Use Newton's method with initial "guess" $x_0 = 2$ for the root and compute the iterates x_1, x_2 and x_3 and also the error in each iterate. Is the rate of convergence about what you would expect?

Problem 9. The polynomial

$$p(x) = x^5 + x + 1$$

has a root in the interval $[-1, 0]$. Use the midpoint of this interval as an initial guess and apply Newton's method once. What is your new approximation to the root?

Problem 10. If $G(x) = 4 + 3\sqrt{x}$ then G maps the interval $I = [4, 49]$ into itself and $|G'(x)| \leq 3/4$ for each $x \in I$. Thus G has a unique fixed point in I. **(A)** Find the fixed point. **(B)** If

$$x_{n+1} = G(x_n),\ \ x_0 = 25$$

does the sequence x_n converge to the fixed point? Why?

Problem 11. If $F(x) = x^2 - 1$ find the fixed point(s) of F. If $x_{n+1} = F(x_n)$ for each n and $x_0 = 0$ does x_n converge to a fixed point of F?

Problem 12. If $G(x) = \frac{1}{2} + \frac{1}{2}\sqrt{x+1}$ then G maps the interval $[0, 3]$ into itself. Find the fixed point(s) of G in $[0, 3]$. If x_0 is any point in $[0, 3]$ and $x_{n+1} = G(x_n)$ can you guarantee that x_n converges to a fixed point of G? Explain.

Problem 13. If $P_n(x)$ is the Taylor polynomial centered at a, of degree $\leq n$, for the function f, then the error in $P_n(x)$, when we view $P_n(x)$ as an approximation of $f(x)$, is

$$f(x) - P_n(x) = \frac{1}{n\,!} \int_a^x (x-t)^n\ f^{(n+1)}(t)\,dt = \frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}(x-a)^{n+1}$$

where ξ_x is some point between the numbers a and x.

Suppose f is a five times continuously differentiable function and $f(1) = 2$, $f'(1) = -2$, $f''(1) = 4$, $f^{(3)}(1) = -8$, and $f^{(4)}(1) = -6$.

Part (A): Find the Taylor polynomial $P_4(x)$ of f of degree ≤ 4 with center at 1.

Part (B): Suppose

$$|f^{(5)}(x)| \leq 4 \ \text{ for each } \ x \in [0, 3].$$

Find an estimate for the error $|f(x) - P_4(x)|$ for $x \in [0, 3]$.

Problem 14. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and let B be a 2×2 matrix. If

$$AB = B + I$$

where I is the 2×2 identity matrix compute B.

Problem 15. Let

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 4 & 4 & 1 \end{bmatrix}.$$

Let B be a 3×3 matrix such that

$$CB = C + B + I$$

where I is the 3×3 identity matrix. Find B.

Problem 16. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. First compute $(A^3 + 2A)^2$. Then compute

$$A\left((A^3 + 2A)^2 + 17I\right)^{1,423,846,679}$$

where I is the 2×2 identity matrix.

Problem 17. Find all values of x such that the matrix

$$B = \begin{bmatrix} 1 & x & 1 \\ x & 1 & x \\ 2 & 3 & 5 \end{bmatrix}$$

is invertible.

Problem 18. Let

$$B = \begin{bmatrix} 289490605321453 & 142915498732717 \\ 142917092622049 & 70555200044678 \end{bmatrix}.$$

Compute the determinant $\det(B)$. If you do this calculation directly on a calculator explain why (or why not) that you may have a problem. **Remark**. The condition number of the matrix is about 1.87×10^{29}.

Problem 19. Compute the determinant of the matrix A given by

$$A = \begin{bmatrix} a & 1 & 2 \\ 1 & b & 3 \\ 2 & 3 & c \end{bmatrix}$$

and simplify. If $a = b = c = 0$ is the matrix A invertible?

Problem 20. Let S be a set of 9 vectors in $\mathbb{R}^8$. Which of the following statements is (always) true?

A.) S is linearly independent

B.) S spans $\mathbb{R}^8$

C.) S is linearly dependent

D.) S does not span $\mathbb{R}^8$

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 20.*

Problem 21. Let S be a set of 7 vectors in $\mathbb{R}^8$. Which of the following statements is (always) true?

A.) S is linearly independent

B.) S spans $\mathbb{R}^8$

C.) S is linearly dependent

D.) S does not span $\mathbb{R}^8$

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 21.*

Problem 22. The matrix

$$B = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$$

is invertible if

A.) $a \neq 1$

B.) $a \neq -1$

C.) $a \neq 0$

D.) never

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 22.*

Problem 23. Consider the linear system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ 2x_1 - 2x_3 &= c \end{aligned}$$

Find the value(s) of c for which the system has a solution.

Problem 24. Find the LU decomposition of the Hilbert matrix

$$H_3 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

Problem 25. Find a Cholesky decomposition $H_2 = C^T C$ of the Hilbert matrix

$$H_2 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

Here C is upper triangular.

Problem 26. Consider the linear system

$$\begin{aligned} 2x_1 + x_2 &= -2 \\ x_1 + 3x_2 &= 1 \end{aligned}$$

Use the initial vector $[0, 0]^T$ and perform 3 Jacobi iterations to obtain an approximate solution. Given the actual solution $\left[-\frac{7}{5}, \frac{4}{5}\right]^T$ compute the 2–norm of the error in the approximate solution you obtained.

Problem 27. Consider the linear system

$$\begin{aligned} 2x_1 + x_2 &= -2 \\ x_1 + 3x_2 &= 1 \end{aligned}$$

Use the initial vector $[0,0]^T$ and perform 3 Gauss-Seidel iterations to obtain an approximate solution. Given the actual solution $\left[-\frac{7}{5}, \frac{4}{5}\right]^T$ compute the 2–norm of the error in the approximate solution you obtained.

Problem 28. Use Newton's method (just two iterations) with initial guess $x_0 = 4$ and $y_0 = -6$ to estimate a solution to the nonlinear system

$$\begin{aligned} x^2 - 2x + y &= 1 \\ x + xy - 2y &= -8 \end{aligned}$$

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Problem 29. *(25 points).* The polynomial

$$x^4 + 5x^3 - 19x^2 - 29x + 42$$

has roots 1 and -2. Find the remaining 2 roots.